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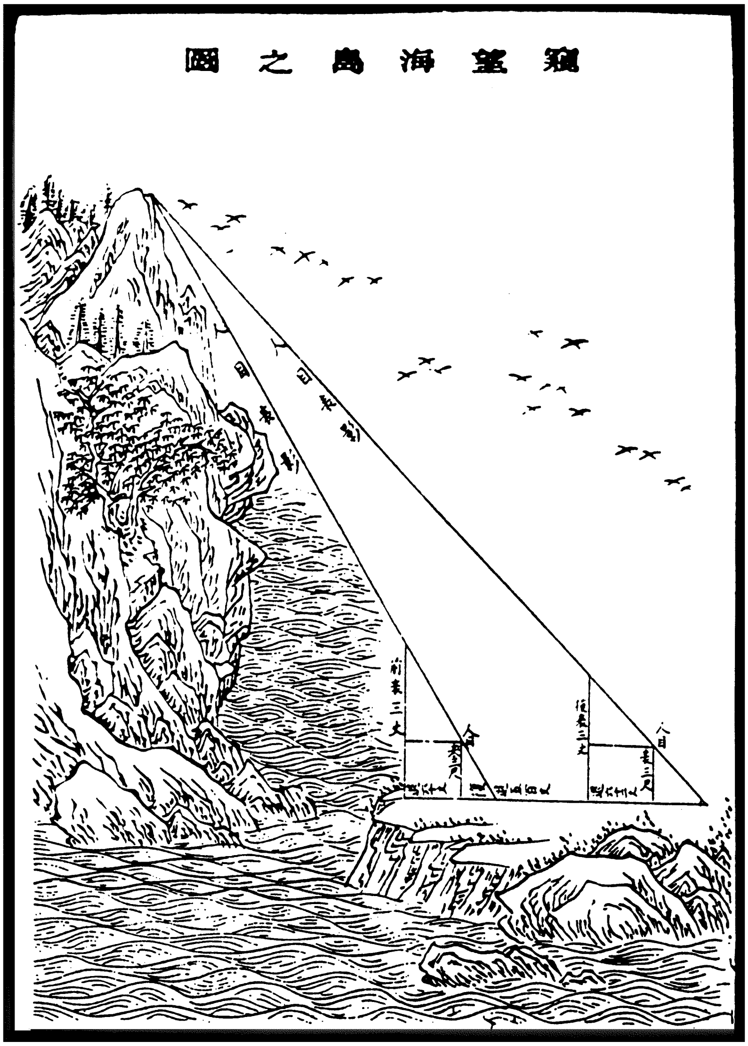
MATHEMATICS MAGAZINE



Measuring an Island Crag (see pp. 163 ff)

- Liu Hui and the First Golden Age of Chinese Mathematics
- Trisection of Angles, Classical Curves, and Functional Equations

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ARTICLES

Liu Hui and the First Golden Age of Chinese Mathematics

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Introduction

Very little is known of the life of Liu Hui, except that he lived in the Kingdom of Wei in the third century A.D., when China was divided into three kingdoms at continual war with one another. What is known is that Liu was a mathematician of great power and creativity. Liu's ideas are preserved in two works which survived and became classics in Chinese mathematics. The most important of these is his commentary, dated 263 A.D., on the *Jiuzhang suanshu*, the great problem book known in the West as the *Nine Chapters on the Mathematical Art*. The second is an independent work on mathematics for surveying, the *Haidao suanjing*, known as the *Sea Island Mathematical Manual*.

In this paper I would like to tell you about some of the remarkable results and methods in these two works. I think they should be more widely known, for several reasons. First, we and our students should know more about mathematics in other cultures, and we are probably less familiar with Chinese mathematics than with the Greek, Indian, and Islamic traditions more directly linked to the historical development of modern mathematics. Second, Western mathematicians who do know something about the Chinese tradition often characterize Chinese mathematics as calculational and utilitarian rather than theoretical. Chinese mathematicians, it is said, developed clever methods, but did not care about mathematical justification of those methods. For example,

> Mathematics was overwhelmingly concerned with practical matters that were important to a bureaucratic government: land measurement and surveying, taxation, the making of canals and dikes, granary dimensions, and so on... Little mathematics was undertaken for its own sake in China. [**2**, p. 26]

While there is justice in this generalization, Liu Hui and his successors Zu Chongzhi and Zu Gengzhi were clearcut exceptions. Their methods were different from those of the Greeks, but they gave arguments of cogency and clarity which we can honor today, and some of those arguments involved infinite processes which we recognize as underlying the integral calculus.

My final reason is that I think mathematical genius should be honored wherever it is found. I hope you will agree that Liu Hui is deserving of our honor.

To understand the context of Liu's work, we must first consider the state of Chinese mathematical computation in the third century A.D. We will then look at the general nature of the *Nine Chapters* and Liu's commentary on it, and at Liu's *Sea Island Mathematical Manual*. I will then focus on three of Liu's most remarkable achievements in geometry—his calculation of π, his derivation of the volume of pyramidal solids, and his work on the volume of a sphere and its completion by Zu Gengzhi.

Chinese Calculation in the First Century A.D.

From at least the period of the Warring States (475–221 B.C.) a base ten positional number system was in common use in China **[12]**. Calculations were done using rods made from bone or bamboo, on a counting board marked off into squares. The numerals from 1 to 9 were represented by rods, as in FIGURE 1. Their placement in squares, from left to right, represented decreasing powers of ten. Rods representing odd powers of ten were rotated 90° for clarity in distinguishing the powers. A zero was represented simply by a blank square, called a *kong*, where the marking into squares prevented the ambiguity sometimes present in, say, the Babylonian number system.

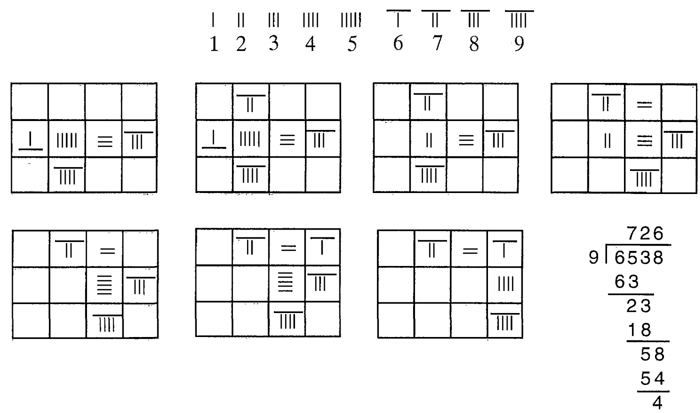


FIGURE 1
Numerals and the division algorithm.

There were efficient algorithms for addition, subtraction, multiplication, and division. For example, the division algorithm is shown in FIGURE 1, except that you should imagine the operations being done rapidly with actual sticks. Notice the close relationship to our modern long division algorithm, although subtraction is easier because sticks are physically removed. In fact, it is identical to the division algorithm given by al-Khwarizmi in the ninth century and later transmitted to Europe, raising the complicated problem of possible transmission through India to the West **[12]**. (See **[17]** for a conservative discussion.)

Notice how the answer $726\frac{4}{9}$ ends up with 726 in the top row, and then 4 above 9. This led Chinese calculators to represent fractions by placing the numerator above the denominator on the counting board. By the time of the *Nine Chapters* there was a completely developed arithmetic of fractions: they could be multiplied, divided, compared by cross multiplication, and reduced to lowest form using the "Euclidean algorithm" to find the largest common factor of the numerator and denominator. Addition was performed as $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$, and then the fraction was reduced if necessary. In the *Nine Chapters*, 160 of the 246 problems involve computations with fractions **[11]**.

We will see that Chapter Eight of the *Nine Chapters* solves systems of linear equations by the method known in the West as "Gaussian Elimination" after C. F. Gauss (1777–1855), which, of course, involves subtracting one row of numbers from another. In the course of such calculations, it is inevitable that negative numbers will arise. This presented no problems to Chinese calculators: two colors of rods were used, and correct rules were given for manipulating the colors. Liu Hui suggested in his commentary on the *Nine Chapters* that negative numbers be treated abstractly:

> When a number is said to be negative, it does not necessarily mean that there is a deficit. Similarly, a positive number does not necessarily mean that there is a gain. Therefore, even though there are red (positive) and black (negative) numerals in each column, a change in their colors resulting from the operations will not jeopardize the calculation. [**17**, pp. 201–202]

Perhaps most remarkably, Chinese mathematicians had developed by the time of the *Nine Chapters* efficient algorithms for computing square roots and cube roots of arbitrarily large numbers. The algorithm for the square root computed the root digit by digit, by the same method which used to be taught in American schools before the coming of the calculator. Martzloff [**17**] works through an example, and Lam [**11**] shows how it would look on a counting board. The algorithm for finding cube roots was similar, although, of course, more complicated.

In other words, by the time of the *Nine Chapters* the Chinese had developed a number system and a collection of calculational algorithms essentially equivalent to our modern system, with the exception of decimal fractions.

Nine Chapters on the Mathematical Art

Nine Chapters on the Mathematical Art is a compilation of 246 mathematical problems loosely grouped in nine chapters. Some of its material predates the great book-burning and burial-alive of scholars of 213 B.C., ordered by emperor Shih Huang-ti of the Qin dynasty. Indeed, Liu Hui writes in the preface of his commentary:

> In the past, the tyrant Qin burnt written documents, which led to the destruction of classical knowledge... Because of the state of deterioration of the ancient texts, Zhang Cang and his team produced a new version... filling in what was missing. [**17**, p. 129]

It is believed that the *Nine Chapters* were put in their final form sometime before 100 A.D. It "became, in the Chinese tradition, the mandatory reference, the classic of classics." [**17**, p. 14] At the time of this writing there is no complete English translation of the *Nine Chapters*, although there are many scholarly Chinese editions, and translations into Japanese, German, and Russian. An English translation by J. N. Crossley and Shen Kangsheng is in preparation, to be published by Springer-Verlag. For summaries, see [**11**], [**17**], [**18**], [**21**].

The format of the *Nine Chapters* is terse: a problem, its answer, and a recipe for obtaining the answer. Usually no justification is given for the method of solution. Just the facts.

Chapter One has many problems on the arithmetic of fractions, and a section on computing areas of planar figures, with correct formulas for rectangles, triangles, and trapezoids. Here's a problem on the area of a circle:

1.32: There is a circular field, circumference 181 *bu* and diameter $60\frac{1}{3}$ *bu*. Find the area of the field.

Answer: 11 *mu* $90\frac{1}{12}$ *bu*. (1 *mu* = 240 *bu*)

Method: Mutually multiply half of the circumference and half of the diameter to obtain the area in *bu*. Or multiply the diameter by itself, then by 3 and divide by 4. Or multiply the circumference by itself and divide by 12. [**11**, p. 13]

The first method is correct, but the data of the problem and the other two methods assume that the ratio of the circumference of a circle to its diameter, which we call π, is three. This assumption is made throughout the *Nine Chapters*.

Chapter Two is a series of commodity exchange problems involving proportions. Chapter Three concerns problems of "fair division." The solutions given may not seem very fair to us:

3.8: There are five persons: Dai Fu, Bu Geng, Zan Niao, Shang Zao, and Gong Shi. They pay a total of 100 *qian*. A command desired that the highest rank pays the least, and the successive ones gradually more. Find the amount each has to pay.

Answer: Dai Fu pays $8\frac{104}{137}$ *qian*; Bu Geng pays $10\frac{130}{137}$ *qian*; Zan Niao pays $14\frac{82}{137}$ *qian*; Shang Zao pays $21\frac{123}{137}$ *qian*; Gong Shi pays $43\frac{109}{137}$ *qian*. [**11**, p. 21]

The method calls for dividing the cost in proportions $\frac{1}{5}:\frac{1}{4}:\frac{1}{3}:\frac{1}{2}:1$, which gives practice in adding fractions, but badly exploits the lowest rank person!

Chapter Four contains problems asking for the calculation of square roots and cube roots. The last problem of Chapter Four is

4.24: There is a sphere of volume 16441866437500 *chi*. Find the diameter.

Answer: 14300 *chi*.

Method: Put down the volume in *chi*, multiply by 16 and divide by 9. Extract the cube root of the result to get the diameter of the sphere. [**11**, p. 23]

This gives the formula $V=\frac{9}{16}d^3$ for the volume of a sphere in terms of its diameter, which isn't correct even if we take $\pi=3$.

Chapter Five asks for the volumes of a number of solids, including several different kinds of pyramids, frustums of pyramids, cones and their frustums, and a wedge with a trapezoidal base. The given formulas are all correct, but no hint is given of how they were derived.

Chapter Six deals with fair division in a much more realistic way than the problems in Chapter Three. There are problems on transporting grain, taxation, and irrigation. There are also some less realistic problems which make one wonder how Chinese students must have felt about "word problems":

6.14: There is a rabbit which walks 100 *bu* before it is chased by a dog. When the dog has gone 250 *bu*, it stops and is 30 *bu* behind the rabbit. If the dog did not stop, find how many more *bu* it would have to go before it reaches the rabbit.

Answer: $107\frac{1}{7}$ *bu*. [**11**, p. 28]

Chapter Seven has a number of problems involving two linear equations in two unknowns, usually solved by the method of "false position." Problems in Chapter Eight involve solving n linear equations in n unknowns for n up to 5. The method of solution, described in detail, is Gaussian elimination on the appropriate matrix represented on the counting board. The Chinese called this method *fangcheng*. See [**17**] for an extended example. Perhaps the most interesting problem is

8.13: There are five families which share a well. 2 of A's ropes are short of the well's depth by 1 of B's ropes. 3 of B's ropes are short of the depth by 1 of C's ropes. 4 of C's ropes are short by 1 of D's ropes. 5 of D's ropes are short by 1 of E's ropes. 6 of E's ropes are short by 1 of A's ropes. Find the depth of the well and the length of each rope.

Answer: The well is 721 *cun* deep. A's rope is 265 *cun* long. B's rope is 191 *cun* long. C's rope is 148 *cun* long. D's rope is 129 *cun* long. E's rope is 76 *cun* long. [**11**, p. 37]

Notice that this problem involves five equations and six unknowns, and thus is indeterminate. Liu Hui pointed out that the solution gives only the necessary proportions for the lengths. It is also the smallest solution in integer lengths.

The problems in Chapter Nine involve right triangles and the "Pythagorean" theorem, which had long been independently known in China, where it was called the *gou-gu* theorem **[26]**. No proof is given of this theorem, or of a correct formula for the diameter of the inscribed circle in a right triangle. Similar right triangles are used to solve surveying problems involving one unknown distance or length.

Liu Hui's Commentary

The *Nine Chapters* presents its solution methods without justification. Liu Hui in his commentary set himself the goal of justifying those methods. One reason was practical, as Liu wrote about the *Nine Chapters'* use of 3 for the ratio of the circumference of a circle to its diameter:

> Those who transmit this method of calculation to the next generation never bother to examine it thoroughly but merely repeat what they learned from their predecessors, thus passing on the error. Without a clear explanation and definite justification it is very difficult to separate truth from fallacy. [**20**, p. 349]

Another reason has to do with seeing and appreciating the logical structure of mathematics:

> Things are related to each other through logical reasons so that like branches of a tree, diversified as they are, they nevertheless come out of a single trunk. If we elucidate by prose and illustrate by pictures, then we may be able to attain conciseness as well as comprehensiveness, clarity as well as rigor. [**20**, p. 355]

In this section, we'll begin our examination of Liu's attempt to attain "clarity as well as rigor" by looking at five of his contributions.

Problems in Chapter Four of the *Nine Chapters* require taking square roots using the square root algorithm. To take the square root of a $2k+1$ or $2k+2$ digit number N, the algorithm begins by finding the largest number $A_0 = a_0 \times 10^k$, where a_0 is a digit, such that $A_0^2 \le N$. Then compute $N_1 = N - A_0^2$. Now find the largest $A_1 = a_1 \times 10^{k-1}$ such that $A_1(2A_0 + A_1) \le N_1$, and form $N_2 = N_1 - A_1(2A_0 + A_1)$. Continue in this manner. If N is a perfect square, its square root will be the $(k+1)$-digit number $S = a_0 a_1 \cdots a_k$.

Liu Hui first gives a geometric argument, similar to arguments used in Greek geometric algebra, to explain why the algorithm works. Consider FIGURE 2, which is not to scale. (Liu's original figures were all lost, but most of them are easy to reconstruct from his verbal descriptions.) From a square of area N, we first subtract a square of side A_0, then the L-shaped figure of width A_1, which the Greeks called a gnomon, then a gnomon of width A_2, and so on until we exhaust the square.

Well, at least we exhaust the square if N is a perfect square, as it is in many of the *Nine Chapters* problems. (Some of the problems involve rational perfect squares, for instance $N = 564752\frac{1}{4}$ in problem 4.15.) But Liu also asks what happens if N is not a perfect square: "In this case it is not sufficient to say what the square root is about by simply ignoring the [remaining] gnomon." [**7**, p. 211] For integral but non-square N,

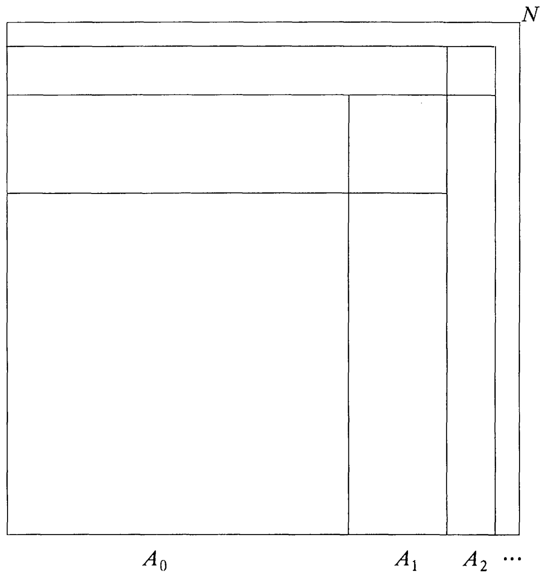


FIGURE 2
Geometry of the square root algorithm.

the square root algorithm yields $N = S^2 + R$, where $0 < R < 2S + 1$. Liu gives two ways of approximating the square root. The first is to take a rational approximation using

$$S + \frac{R}{2S+1} < \sqrt{N} < S + \frac{R}{2S}. \quad \textbf{[17]}$$

The second is even more interesting. If we continue the algorithm on the counting board past the last digit of N, we get

$$\sqrt{n} \approx a_0 a_1 \dots a_k + \frac{a_{k+1}}{10} + \frac{a_{k+2}}{100} + \dots .$$

The ancient Chinese had names for the fractions $1/10^k$ for k up to five. Liu suggests continuing the calculation down to "those small numbers for which the units do not have a name," and if necessary adding a fraction to a_{k+5} to get even greater accuracy **[11]**. In other words, it is not stretching very much to say that Liu Hui invented decimals; he certainly invented their calculational equivalent. We will see that he needed this kind of accuracy for his calculation of π. Liu also gave a justification for the cube root algorithm using a three-dimensional figure similar to FIGURE 2.

Chapter Eight of the *Nine Chapters* solved systems of linear equations using the *fangcheng* method on a counting board matrix: multiples of rows (actually columns, since the equations were set up vertically on the counting board) were systematically subtracted from other rows to reduce the matrix to triangular form. Liu Hui explains that the goal of this method is to reduce to a minimum the number of computations needed to find the solution: "generally, the more economic a method is, the better it is." In fact, Liu compares two different *fangcheng* methods for solving problem 8.18 by counting the number of counting board operations needed in each method **[17]**. Surely this is the first example in history of an operation count to compare the computational efficiency of two algorithms.

Finally, Chapter Nine of the *Nine Chapters* presented, without justification, solutions to a number of problems involving right triangles. Liu Hui justified these solutions by a series of ingenious "dissection" arguments, based on the principles that congruent figures have the same area, and that if we dissect a figure into a finite number of pieces, its area is the sum of the areas of the pieces. I'll give two examples.

The solution to problem 9.16 finds the diameter d of a circle inscribed in a right triangle with legs a and b and hypotenuse c by

$$d = \frac{2ab}{a+b+c}.$$

Liu's dissection proof of this result can be reconstructed as in FIGURE 3 [**20**]. See it?

For the second example, consider the famous *gou-gu* theorem that for a right triangle as above, $a^2 + b^2 = c^2$. For this theorem, Liu's verbal description of his proof is as follows:

> The shorter leg multiplied by itself is the red square, and the longer leg multiplied by itself is the blue square. Let them be moved about so as to patch each other, each according to its type. Because the differences are completed, there is no instability. They form together the area of the square on the hypotenuse. [**31**, p. 71]

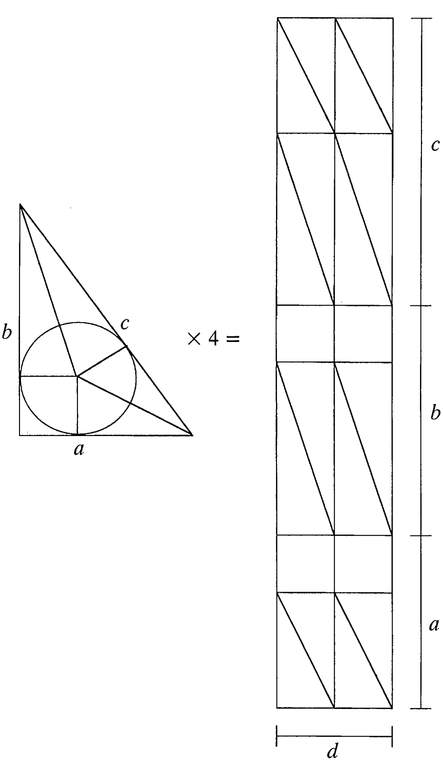
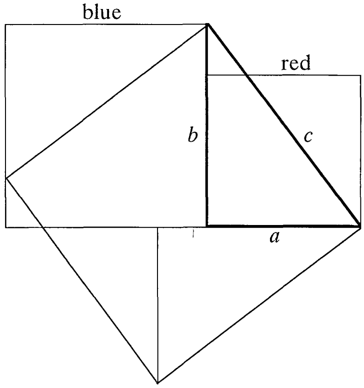


FIGURE 3
Diameter of a circle inscribed in a right triangle.

Clearly, Liu had a dissection proof of the *gou-gu* theorem. Just as clearly, the verbal description does not enable us to reconstruct Liu's diagram. FIGURE 4 shows two



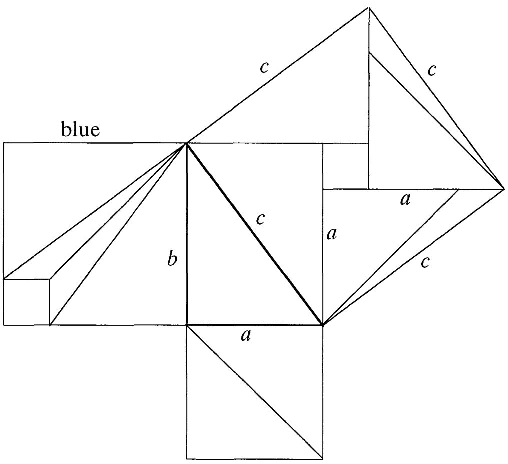


FIGURE 4
Dissection proofs of the *gou-gu* theorem.

proposed constructions. The first, where the square on the hypotenuse is allowed to overlap the squares on the legs, is due to Gu Guanguang in 1892, reported in **[17]**. The second, less straightforward but without overlapping squares, is from **[31]**.

The Sea Island Mathematical Manual

Chapter Nine of the *Nine Chapters* included surveying problems involving one unknown distance or length. However, most real surveying problems involve several such unknowns. For example, we might wish to determine the height of, and distance to, a mountain which is inaccessible, perhaps because it is on an island we cannot reach. Liu Hui pointed out that we can do this by making two observations, and worked out the geometry of how to make two observations yield the unknown distances. If we wish also to know the height of a pine tree on top of that inaccessible mountain, we can do it with three observations. His compilation of solutions to nine illustrative surveying problems became the *Sea Island Mathematical Manual*. The mountain on the sea island is the first problem; the pine tree is the second. **[1]** and **[24]** include complete translations with commentary.

Here is the sea island problem:

> For looking at a sea island, erect two poles of the same height, 30 *chi*, the distance between the front and rear pole being 6000 *chi*. Assume that the rear pole is aligned with the front pole. Move away 738 *chi* from the front pole and observe the peak of the island from ground level; it is seen that the tip of the front pole coincides with the peak. Move backward 762 *chi* from the rear pole and observe the peak from ground level again; the tip of the rear pole also coincides with the peak. What is the height of the island and how far is it from the front pole?
>
> Answer: The height of the island is 7530 *chi*. It is 184500 *chi* from the front pole. [**24**, p. 20]

The extant version of the *Sea Island Manual* contains only the problems, answers, and recipes for obtaining the answers, exactly as in the *Nine Chapters*. Liu Hui also gave proofs for the correctness of his methods, but these proofs and the accompanying diagrams were not preserved, and the best we can do is offer plausible reconstructions. Using the notation of FIGURE 5, Liu's method for solution corresponds to the formulas

$$h = x + b = \frac{bd}{a_1 - a_2} + b, \qquad y = \frac{a_2 d}{a_1 - a_2}.$$

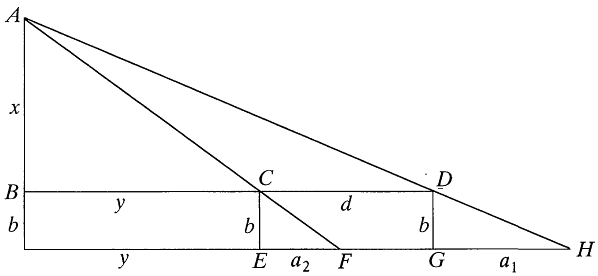


FIGURE 5
The height of a sea island.

We must obtain these formulas using only similar right triangles, since there was no concept of angle, much less any trigonometry, in ancient Chinese mathematics, nor was there any use of similar triangles other than right triangles. Here is one method. Since $\Delta ABD \sim \Delta DGH$,

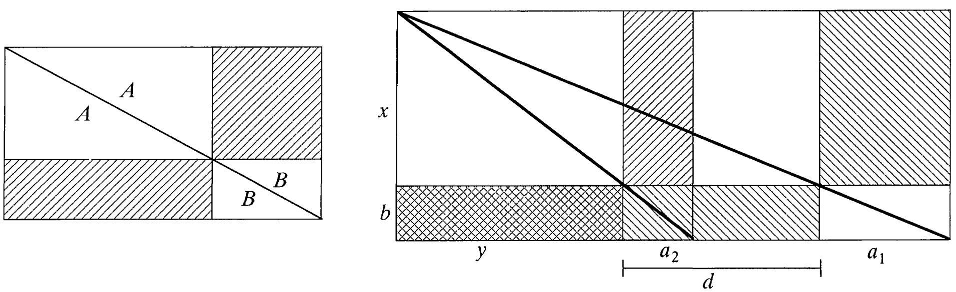
$$\frac{x}{y+d} = \frac{b}{a_1}, \quad \text{so } xa_1 = by + bd. \tag{1}$$

Since $\Delta ABC \sim \Delta CEF$,

$$\frac{x}{y} = \frac{b}{a_2}, \quad \text{so } xa_2 = by. \tag{2}$$

Subtracting these equations gives $x(a_1 - a_2) = bd$, which leads to the expression for the height, and then substitution gives the distance.

Swetz **[24]** gives a very plausible alternate derivation which avoids the use of similar triangles completely. It is based on a lemma about rectangles which is illustrated in FIGURE 6a: if we divide a rectangle into four smaller rectangles at any point on its



(*a*) A rectangular lemma. (*b*) A rectangular proof.

FIGURE 6

diagonal, then the two rectangles shaded in the figure must have the same area. This follows from a dissection argument. The diagonal divides the rectangle into two congruent triangles. From these triangles, subtracting the congruent triangles labeled A and B yields the given rectangles. If we apply this result twice to FIGURE 6b, the equal \\\ rectangles give equation (1), and the equal /// rectangles give equation (2). This method is also discussed in [**9**].

The *Sea Island Manual* was certainly not the deepest mathematics which Liu Hui did, but it probably had the greatest immediate impact. Recall that the kingdom of Wei was continually at war during the time of Liu's work. Surveying was important for maps which supported war, as well as the administrative bureaucracy. Needham reports that the Wei general Deng Ai always "estimated the heights and distances, measuring by finger breadths before drawing a plan of the place and fixing the position of his camp." [**24**, p. 15] There is an interesting parallel in the West. Swetz notes that Greek armies had a specific reason for wanting to calculate unknown height at an inaccessible distance, quoting Heron of Alexandria:

> How many times in the attack of a stronghold have we arrived at the foot of the ramparts and found that we made our ladders and other necessary implements for the assault too short, and have consequently been defeated simply for not knowing how to use the Dioptera for measuring the heights of walls; such heights have to be measured out of the range of enemy missiles. [**24**, p. 28]

The Calculation of π

Recall that problem 1.32 of the *Nine Chapters* gave the correct formula for the area of a circle, but used a value of three for π. Liu points out that for a circle of radius one, the area of a regular dodecagon inscribed in the circle is three, so the area of the circle must be greater than three. He then proceeds to estimate the area of the circle more exactly by calculating the areas of inscribed $3 \cdot 2^n$-gons as follows. In a circle of radius r, let c_n be the length of the side of an inscribed n-gon, a_n be the length of the perpendicular from the center of the circle to the side of the n-gon, and S_n be the area of the n-gon. See FIGURE 7. Then we can calculate inductively

$$\begin{aligned} c_6 &= r, \\ a_n &= \sqrt{r^2 - (c_n/2)^2}, \\ c_{2n} &= \sqrt{(c_n/2)^2 + (r - a_n)^2}, \\ S_{2n} &= \frac{1}{2} n r c_n. \end{aligned}$$

The last formula is clever, and follows from noticing that each of the $2n$ triangles making up the $2n$-gon can be thought of as having base r and height $c_n/2$. Moreover, FIGURE 7 shows that the area S of the circle satisfies

$$S_{2n} < S < S_n + 2(S_{2n} - S_n) = 2S_{2n} - S_n.$$

Liu considers what happens when we take n larger and larger: "the finer one cuts, the smaller the leftover; cut after cut until no more cut is possible; then it coincides with the circle and there is no leftover." [**20**, p. 347] As n gets large, S_{2n} approaches the area of the circle and nc_n approaches the circumference, so we have justified the *Nine Chapters* claim that the area of a circle is one-half the product of its radius and circumference.

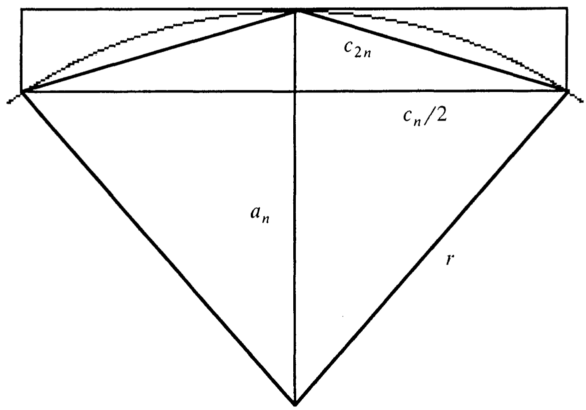


FIGURE 7
The calculation of π.

Taking $r = 10$, Liu Hui carries out the calculations, keeping 6-place accuracy, up to $n = 96$, hence approximating the circle by a 192-gon. He concludes that

$$3.1410 < \pi < 3.1427,$$

and suggests that for practical calculations it should be enough to use $\pi \approx 3.14$. Either Liu or some interpolating later commentator carried the computation as far as $n = 1536$ and obtained the approximation $\pi = 3.1416$. See **[13]** and **[28]** for treatments of the intricacies of this kind of calculation. **[13]** gives a translation of Liu Hui's text.

If we compare this treatment to Archimedes' in *Measurement of a Circle*, the similarities are striking, although the differences are also interesting. Archimedes, of course, included a formal proof by the method of exhaustion required by the conventions of Greek geometry. However, the subdivision method and the inductive calculation are essentially the same. Archimedes obtained his upper bound by considering circumscribed polygons, instead of Liu's clever method of using only inscribed polygons. Archimedes used 96-gons to obtain his famous estimate

$$3\frac{10}{71} < \pi < 3\frac{1}{7}, \quad \text{or} \quad 3.1409 < \pi < 3.1428.$$

Two centuries later Zu Chongzhi (429–500 A.D.) carried Liu Hui's approach farther. Using a polygon of 24576 sides, Zu obtained the bounds $3.1415926 < \pi < 3.1415927$. See **[13]** and, for a different view, **[28]**. In addition, Zu recommended two rational approximations for π: Archimedes' value of $22/7$, and the remarkably accurate $355/113 \approx 3.1415929$.

Zu's method for arriving at his rational approximation $\frac{355}{113}$ for π is not known. One line of reasoning would be to start with Zu's value of 3.1415926 and the approximation $\frac{22}{7} = 3\frac{1}{7} \approx 3.1428571$, which is slightly too large, and ask for a fraction which, when added to 3, would give a better approximation than $\frac{1}{7}$ does. It is easy to see that the fractions we should check are those of the form $\frac{k}{7k+1}$. We then try to find k so that

$$\frac{1}{7} - \frac{k}{7k+1} \approx .1428571 - .1415926 = .0012645,$$

$$\frac{1}{49k+7} \approx .0012645, \qquad 49k + 7 \approx 791.$$

The solution $k = 16$ gives the rational approximation $3\frac{16}{113} = \frac{355}{113}$. For another possible approach, see **[17]**.

Zu Chongzhi's approximation of π was not bettered until al-Kashi of Samarkand computed π to 14 decimal places in the early 15th century. The rational approximation 355/113 was not discovered in Europe until the late 16th century.

The Volume of Pyramids

Chapter Five of the *Nine Chapters* gives correct formulas for the volumes of a number of pyramidal solids. For example, the volume of the *chu-tung*, a truncated rectangular pyramid illustrated in FIGURE 11, is correctly given as

$$\frac{h}{6}(2ab + ad + bc + 2cd).$$

Did you know that formula? From it follows the volume of a rectangular pyramid (put $c = d = 0$), a truncated square pyramid (put $a = b$, $c = d$), and a rectangular wedge (put $d = 0$).

Liu Hui gives justifications for these formulas based on dissection arguments and a remarkable limit argument. I will mostly follow the translation and discussion in **[30]**. Liu's argument uses three special solids: a *qiandu*, which is a triangular prism, a *yangma*, which is a rectangular pyramid whose vertex is above one corner of its base, and a *bienao*, which is a tetrahedron with three successive perpendicular edges. See FIGURES 8, 9, and 10.

Liu starts with the case of a cube, which he dissects into three congruent *yangma*, to conclude that the volume of a regular *yangma* is 1/3 the volume of the cube. See FIGURE 8. Since a *yangma* and a *bienao* fit together to make a *qiandu*, which is 1/2 of the cube, the volume of the *bienao* must be 1/6 the volume of the cube. Alternatively, we could get this result by dissecting the *yangma* into two congruent *bienao*.

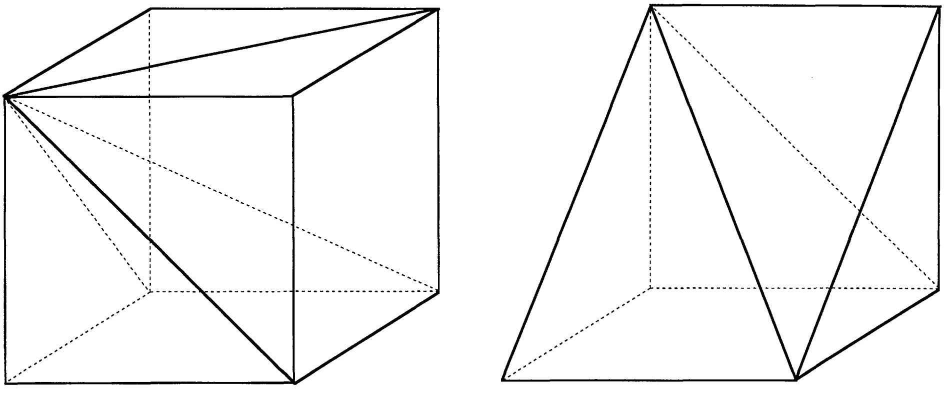


FIGURE 8
Dissecting a cube and a *qiandu*.

Now suppose that instead of a cube, we start with an $a \times b \times c$ rectangular box. We can still dissect it into three *yangma*, but now these *yangma* will have 3 different shapes, so it is not clear that their volumes are equal. We can also dissect a *yangma* into two *bienao*, or assemble a *bienao* and a *yangma* to make a *qiandu*, but again, the *bienao* have 3 different shapes, and it is not clear that their volumes are equal.

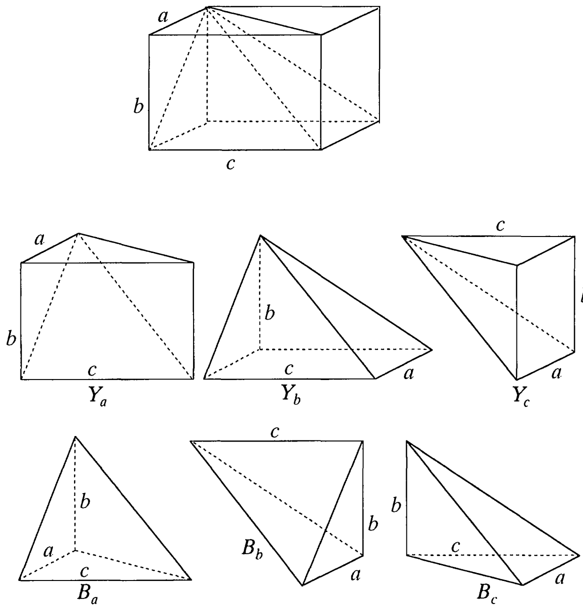


FIGURE 9
Three types of *yangma* and *bienao*.

Using the notation in FIGURE 9, what the dissections do show is that

$$Y_a + Y_b + Y_c = abc$$

$$Y_a + B_a = abc/2 \qquad Y_a = B_b + B_c$$

$$Y_b + B_b = abc/2 \qquad Y_b = B_a + B_c$$

$$Y_c + B_c = abc/2 \qquad Y_c = B_a + B_b.$$

However, this does not give enough information to evaluate the volumes.

Liu proceeds to prove that $Y_b = 2B_b$ (and similarly $Y_a = 2B_a$, $Y_c = 2B_c$), which does allow us to conclude that the volume of each *yangma* is $abc/3$ and that of each *bienao* is $abc/6$. His method is shown in FIGURE 10. Dissect Y_b at the midpoints of its sides into a rectangular box, 2 *qiandu*, and two half-size copies of Y_b (call them Y_b'). Similarly, dissect B_b into 2 *qiandu* and 2 half-size copies of B_b (call them B_b'). Since the box and 2 *qiandu* have twice the volume of 2 *qiandu*, we only need to show that

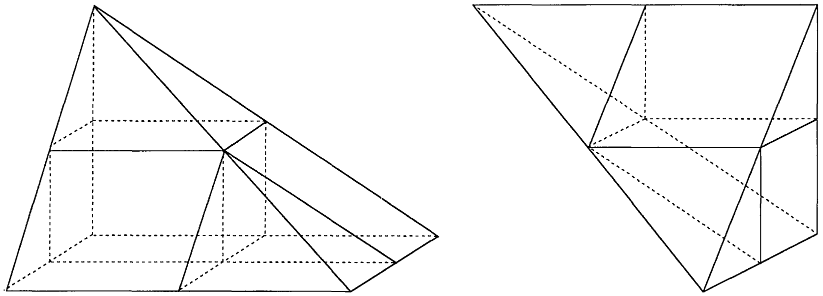


FIGURE 10
Dissecting a *yangma* and a *bienao*.

$Y_b' = 2B_b'$. Liu notes that these new figures together have $1/4$ the volume of the original figures, since the two small *yangma* and *bienao* fit together to form two *qiandu* whose total volume is $abc/8$. Repeat the dissection on each of the new figures, and continue. At each stage the volume we have not yet accounted for is $1/4$ that of the previous stage. Liu expresses what happens in the limit as follows:

> The smaller they are halved, the finer are the remaining dimensions. The extreme of fineness is called minute. That which is minute is without form. When it is explained in this way, why concern oneself with the remainder? [**30**, p. 173]

This is not a modern limit argument, of course. Liu seems to be saying that if we cut the figures into smaller and smaller pieces, we will come to a point where the pieces are so small that they no longer have form or volume. (The terms translated as 'minute' and 'form' are philosophical terms from the *Tao Te Ching*.) Still, we recognize the limit idea, and the recursive dissection argument has a delightful elegance. For some of the philosophical issues, see [**7**], [**16**], and [**30**]. For a comparison to the Greek proof in Euclid's *Elements*, see [**4**].

Knowing the volume of a *yangma*, we can now derive the volumes of the other solids by dissection. For example, let's verify the formula for the volume of the *chu-tung*. Dissect it as in FIGURE 11 into a box L, four *qiandu* of two different shapes Q_a and Q_b, and four *yangma* Y. If we do this to six copies of the *chu-tung*, we have

$$6L + 12Q_a + 12Q_b + 24Y.$$

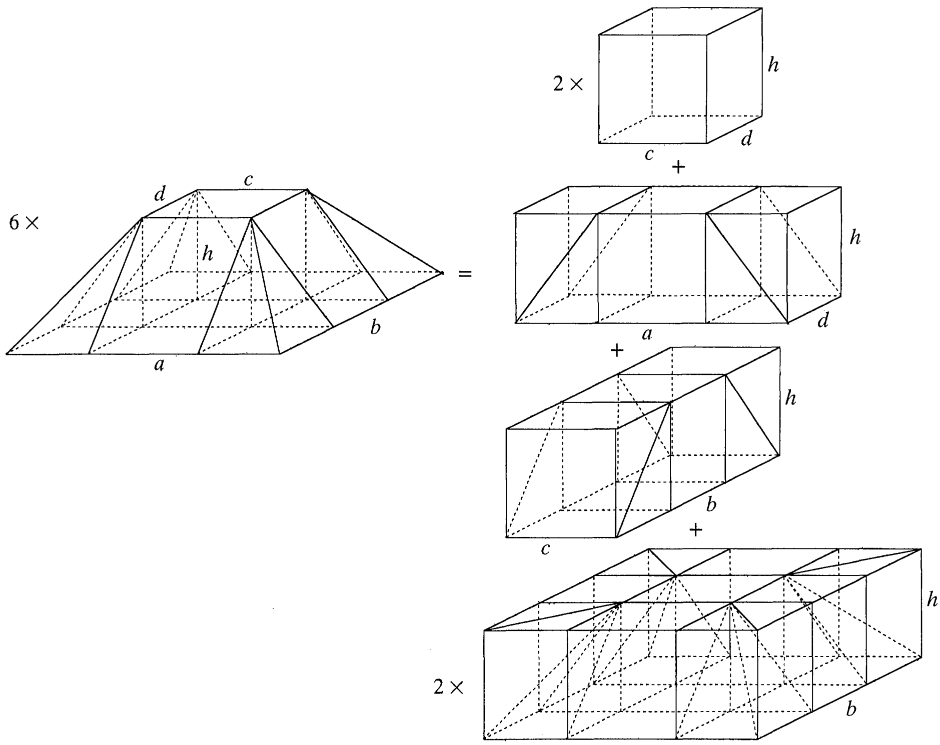


FIGURE 11
The volume of a *chu-tung*.

Now reassemble these, as in FIGURE 12, into

$$\begin{aligned} \text{two boxes of volume } hcd&: \quad 2L \\ \text{one box of volume } had&: \quad L + 4Q_b \\ \text{one box of volume } hbc&: \quad L + 4Q_a \\ \text{two boxes of volume } hab&: \quad 2L + 8Q_a + 8Q_b + 24Y. \end{aligned}$$

Notice that for the last step we need to replace some of the Y_h *yangma* with *yangma* of other shapes, but this is allowable since we have shown that these *yangma* all have the same volume.

Finally, Liu derives the volume of a cone from the volume of a square pyramid, and the volume of a truncated cone from the volume of a truncated square pyramid, by using what we know as "Cavalieri's principle," after Bonaventura Cavalieri (1598–1647). We can state this principle as follows:

> The volumes of two solids of the same height are equal if their planar cross-sections at equal heights always have equal areas; if the areas of the planar cross-sections at equal heights always have the same ratio, then the volumes of the solids also have this ratio.

Liu inscribes the truncated cone, for example, in a truncated square pyramid of the same height, and then says that since each cross-section consists of a circle inscribed in a square, the ratio of the volumes of the truncated cone to the truncated pyramid must be in the same ratio as the area of a circle to its circumscribed square, i.e., $\pi/4$ [7].

The Volume of a Sphere

Recall that problem 4.24 of the *Nine Chapters* gave the volume of a sphere as $\frac{9}{16}d^3$. Liu points out that this is incorrect, even using the inaccurate value of 3 for π. He explains the error as follows. Let a cylinder be inscribed in a cube of side d, and consider the cross-section of this figure by any plane perpendicular to the axis of the cylinder. The plane will cut the cylinder in a circle of diameter d, inscribed in a square of side d. The ratio of these areas is $\pi/4$. Since this is true for each cross-section, the same ratio must hold for the volumes, so that the volume of the cylinder is $\frac{\pi}{4}d^3$. Now consider the sphere of diameter d inscribed in the cylinder. If we assume, incorrectly, that the ratio of the volume of the sphere to the volume of the cylinder is also $\pi/4$, then we get that the volume of the sphere is $\frac{\pi^2}{16}d^3$, which is the *Nine Chapters* result (using $\pi = 3$).

How do we know that the ratio of the volumes of the sphere and cylinder cannot be $\pi/4$? Liu's ingenious argument is as follows. Inscribe a second cylinder in the cube, with axis orthogonal to that of the first cylinder, and consider the intersection of these two cylinders. Liu called this intersection a "double box-lid." See FIGURE 12. Since the sphere is contained in both cylinders, it is contained in the box-lid. Moreover, consider any cross-section of this figure by a plane perpendicular to the axis of the box-lid. The cross-section of the sphere will be a circle, inscribed in the square which is the cross-section of the box-lid, so again the ratio of the areas is $\pi/4$, and since this is true for all cross-sections, the ratio of the volumes of the sphere and the box-lid must also be $\pi/4$. Now the box-lid is certainly smaller than the original cylinder, so the ratio of the volumes of the sphere and the cylinder must be strictly less than $\pi/4$.

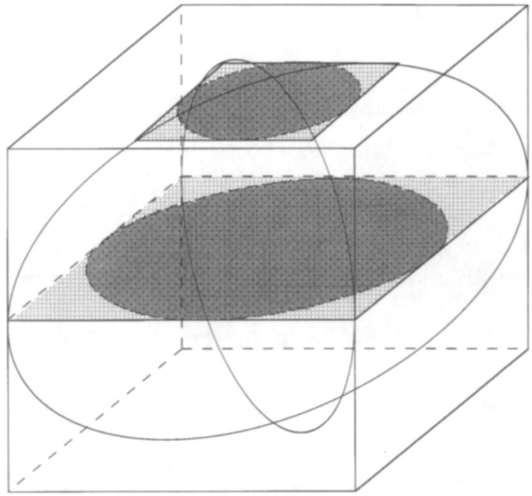


FIGURE 12
Cross sections of a sphere in a double box-lid in a cube.

This lovely argument using Cavalieri's principle shows that the *Nine Chapters* formula is wrong, but in order to use it to find the correct volume of the sphere, we would need to be able to find the volume of the double box-lid. Liu tried to do this, but could not. He recorded his failure in a poem, translated by D. B. Wagner as "The Geometer's Frustration:"

Look inside the cube
And outside the box-lid;
Though the diminution increases,
It doesn't quite fit.

The marriage preparations are complete;
But square and circle wrangle,
Thick and thin make treacherous plots,
They are incompatible.

I wish to give my humble reflections,
But fear that I will miss the correct principle;
I dare to let the doubtful points stand,
Waiting for one who can expound them. [**29**, p. 72]

The wait turned out to be two centuries, and the person Liu waited for was Zu Gengzhi, the son of Zu Chongzhi. Stories associated with Zu Gengzhi are reminiscent of those told about Archimedes and many mathematicians since then. For instance, "he studied so hard when he was still very young that he did not even notice when it thundered; when he was thinking about problems while walking he bumped into people." [**15**, p. 82]

Zu Gengzhi argues as follows. Consider one eighth of the double box-lid inscribed in the cube of side $r = d/2$. See FIGURE 13. If a plane is passed through this figure at height h, it intersects the cube in a square of side r, and the box-lid in a square of side s. By the *gou-gu* theorem, $r^2 - s^2 = h^2$. Hence the area of the gnomon *outside* the box-lid is h^2.

Now Zu Gengzhi considers another solid of height r whose cross-section at height h is h^2: an inverted *yangma* cut from a cube of side r. See FIGURE 13. The part of the cube outside the box-lid, and this *yangma*, have all their corresponding cross-sections of the same area. Zu then states his version of Cavalieri's principle in verse:

If volumes are constructed of piled up blocks [areas],
And corresponding areas are equal,
Then the volumes cannot be unequal. [**29**, p. 75]

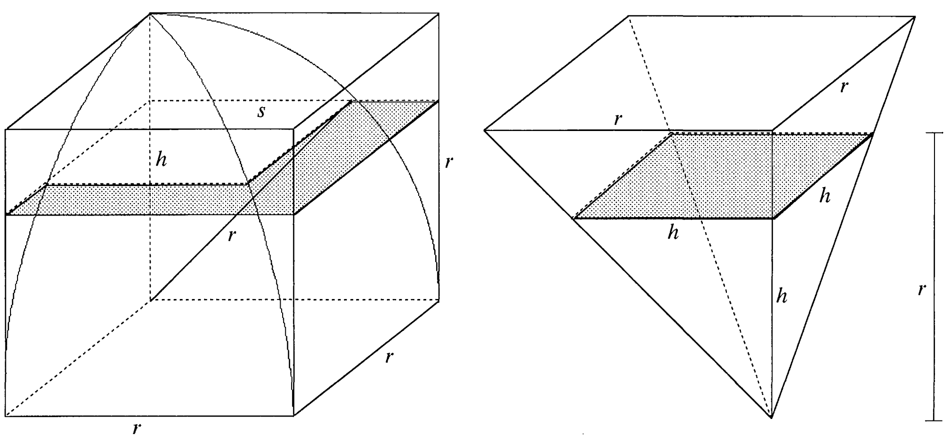


FIGURE 13
The volume outside a box-lid is Cavalieri-equivalent to a *yangma*.

Since the volume of the *yangma* is $\frac{1}{3}r^3$, and the volume outside the box-lid must be the same, the volume inside the box-lid must be $\frac{2}{3}r^3$. Putting the eight pieces together, we get that the volume of the complete double box-lid must be two-thirds of the cube containing it, $\frac{2}{3}d^3$. Remembering Liu Hui's result that the sphere takes up $\pi/4$ of the double box-lid, we finally get the correct formula for the volume of a sphere of diameter d:

$$V = \frac{\pi}{4}\frac{2}{3}d^3 = \frac{\pi}{6}d^3.$$

Following Liu, Zu ends his discussion with a poem, "The Geometer's Triumph:"

The proportions are extremely precise,
And my heart shines.
Chang Heng copied the ancient,
Smiling on posterity;
Liu Hui followed the ancient,
Having no time to revise it.
Now what is so difficult about it?
One need only think. [**29**, pp. 76–77]

One could argue that Liu Hui did not use the full power of Cavalieri's principle, since he only applied it to the situation of one figure inside another, where the cross-sections were circles inscribed in squares. But certainly Zu Gengzhi gave a clear statement of the principle and used its power more than a millennium before Cavalieri [**14**].

There was another precursor, of course. Archimedes had calculated the volume of a sphere, and in Proposition 15 of *The Method,* he calculated the volume of the perpendicular intersection of two cylinders of the same radius. The argument for Proposition 15 is in the part of *The Method* which has not survived, but it is not difficult to reconstruct the reasoning from other demonstrations earlier in the book. Archimedes thought of volumes as made up of planar slices and balanced them on a lever against the slices of other volumes. It is an extension of Cavalieri's principle. For a general discussion of the use of versions of Cavalieri's principle in Greek geometry, see **[10]**.

Conclusion

After the theoretical phase of Chinese mathematics in the 3rd through 5th centuries, represented by Liu Hui, Zu Chongzhi, and Zu Gengzhi, proofs and justifications began to be less important. Although the work of Liu Hui was still taught in the official School for the Sons of the State, instruction began to emphasize rote learning of methods rather than justifications. Liu's diagrams from the commentary on the *Nine Chapters* and arguments from the *Sea Island Manual*, and Zu Chongzhi's work, were lost. The next, brief flowering of creative mathematics in China did not happen until the 13th century, with mathematicians like Qin Jiushao, Li Zhi, Zhu Shijie, and Yang Hui. After the thirteenth century, Chinese mathematics declined again until the period of contact with the West.

It is interesting to speculate why Chinese mathematics, with such a powerful calculational base and such a strong theoretical start, did not develop a coherent, ongoing mathematical tradition. Martzloff **[17]** and Swetz **[25]** review a number of possible reasons: emphasis on practical applications, rote learning, and reverence for established ideas which stifled creativity, uneven state support, and low social status accorded to mathematicians compared to scholars in the humanities.

Nevertheless, the remarkable achievements of Chinese mathematics in its first golden age are worthy of our interest and admiration.

Acknowledgment. I wish to thank the mathematics department of the University of Colorado at Boulder for their hospitality during the writing of this paper, and Victor Katz, Ranjan Roy, and Frank Swetz for suggestions which have improved its quality.

Note. **[8]** and **[21–27]** contain very accessible introductions to Chinese mathematics. **[15]** and **[17]** are comprehensive modern histories of Chinese mathematics which make extensive use of Chinese research. **[18]** and **[19]** are older histories which are still good reading.

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Trisection of Angles, Classical Curves, and Functional Equations

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1. Introduction

The old problem of trisecting angles by means of ruler and compass, unsolvable in its original form, has generated, throughout the history of mathematics, many interesting contributions, including special "tools" to be used either for solving the problem exactly or for finding ingenious approximations (see, e.g., [5]). Contrary to popular belief, several Greek geometers considered tools other than straightedge and compass for this purpose. Among these tools were the spiral of Archimedes (ca. 225 BC), the quadratrix of Hippias (ca. 425 BC), and the conchoid of Nicomedes (ca. 240 BC). All these curves, originally given as geometrical loci, can be considered, using today's language, as examples of solutions of elementary functional equations, with metric equalities defining the curves as loci.

Our aim in this paper is to review these curves from a functional equations point of view and to see to what extent the trisecting property, which they all have, characterizes them. In doing so, we find families of trisecting curves that we have not encountered elsewhere.

This note also has a pedagogical purpose. The interest and beauty of working on geometrical problems by means of functional methods strengthens the link between calculus and geometry.

2. The Archimedean Spiral

Archimedes (ca. 225 BC) introduced his celebrated spiral for the purposes of squaring the circle and multisecting angles. We can describe the Archimedean spiral easily using polar coordinates:

$$r = a\cdot\theta, \tag{1}$$

with a a constant of proportionality (see FIGURE 1).

Given a circle of center O and radius a, the distance from a point P of the spiral to the center equals the length of the arc $\widehat{AB}$ on the circumference corresponding to the central angle θ. So Archimedes was able to multisect any angle AOB by dividing (with euclidean tools) the segment OP into n equal parts $\overline{OP_1}$, $\overline{P_1P_2}, \dots, \overline{P_{n-1}P_n}$, where $P_n = P$, tracing the circles with O as center and $\overline{OP_1}$, $\overline{OP_2}, \dots, \overline{OP_{n-1}}$ as radii and cutting the circles with the spiral at the points $T_1, T_2, \dots, T_{n-1}$. Then $OT_1, OT_2, \dots, OT_{n-1}$ divide the initial angle $\angle AOB$ into n equal parts (in FIGURE 1, $n = 3$).

Note that the point Q of the spiral $r = a\theta$ obtained when $\theta = \pi/2$ is such that $\overline{OQ} = a\cdot\pi/2$, so the rectangle of base $2a$ and height $\overline{OQ}$ has area πa^2; from this, the squaring of the circle of radius a is obtained.

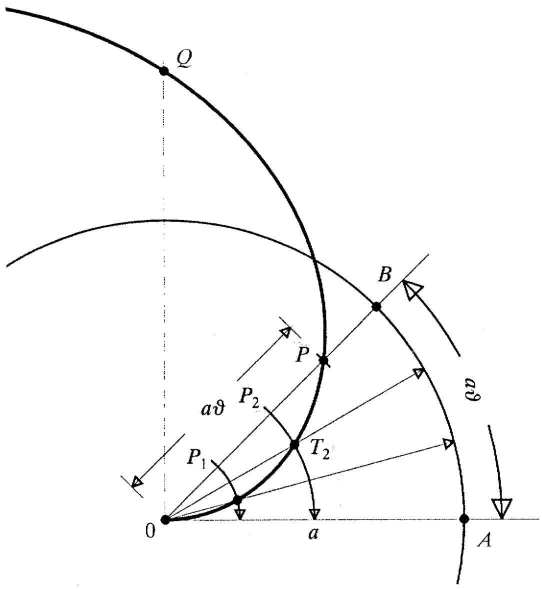

FIGURE 1
An Archimedean spiral trisecting an angle.

Note also that the sum of two Archimedean spirals $r = a\theta$ and $r = b\theta$ is another Archimedean spiral: $a\theta + b\theta = (a+b)\theta$. This has an interesting geometrical meaning: the problem of squaring the circle of radius $a + b$ could be solved either by producing the spiral $r = (a+b)\theta$ or by using the spirals $r = a\theta$ and $r = b\theta$. The following definition is motivated by the preceding properties of the spiral. (In what follows, $\mathbb{R}_+$ denotes the set of nonnegative real numbers.)

DEFINITION. *For fixed $n \geq 2$, an n-spiral function is a function F: $\mathbb{R}_+^2 \to \mathbb{R}_+$ satisfying the following conditions for all $a, b, \theta \geq 0$:*

$$\text{(i)}\quad F(a+b,\theta) = F(a,\theta) + F(b,\theta);\quad \text{(ii)}\quad F\left(a,\frac{\theta}{n}\right) = \frac{1}{n}F(a,\theta).$$

The following theorem characterizes n-spiral functions.

THEOREM 1. *Given $n \geq 2$, a function F: $\mathbb{R}_+^2 \to \mathbb{R}_+$ is an n-spiral function if and only if F can be written in the form*

$$F(a,\theta) = a\varphi(\theta), \tag{2}$$

where $\varphi(0) = 0$ and φ is a solution of the functional equation

$$\varphi\left(\frac{x}{n}\right) = \frac{\varphi(x)}{n} \quad \text{for } x \in \mathbb{R}_+. \tag{E_n}$$

Note. The general solution of (E_n) can be obtained by defining $\varphi(x)$ arbitrarily on the interval $[1, n)$, and extending to $\mathbb{R}_+$ by repeated use of (E_n) itself (see **[8]**).

Proof. If F is an n-spiral function then, by condition (i), for fixed θ the function $f(a) = F(a,\theta)$ satisfies the classical Cauchy equation $f(a+b) = f(a) + f(b)$. Since f is bounded from below by 0 on its domain, it follows that $f(a) = f(1)a$ (for a proof of this implication, see **[1]**). If we let θ vary again we will obtain that $F(a,\theta) = F(1,\theta)\cdot a$, i.e., equation (2) holds with $\varphi(\theta) = F(1,\theta)$. By (ii), φ must satisfy $\varphi(0) = 0$ and (E_n). The converse is obvious. □

COROLLARY 2. *Let F be an n-spiral function, represented in the form* (2). *Then, for any fixed $a > 0$, the plane curve given in polar coordinates by $r = F(a, \theta)$ solves the n-multisection of any angle.*

COROLLARY 3. *Suppose that F is an n-spiral function both for $n = 3$ and for $n = 5$, and that F is continuous or monotonic. Then F is the Archimedean spiral $F(a, \theta) = a\theta$.*

Proof. Theorem 1 and the facts that

$$\varphi\left(\frac{\theta}{3}\right) = \frac{1}{3}\varphi(\theta) \quad \text{and} \quad \varphi\left(\frac{\theta}{5}\right) = \frac{1}{5}\varphi(\theta)$$

imply that $\varphi(3^n 5^m \theta) = 3^n \cdot 5^m \varphi(\theta)$ for all $n, m \in \mathbb{Z}$. Since the set $\{3^n \cdot 5^m \mid n, m \in \mathbb{Z}\}$ is dense in $\mathbb{R}_+$ (see, e.g., **[11]** for a proof of this fact), the corollary follows from the continuity or monotonicity of F. □

Thus we see that a large variety of spirals can be used to trisect any angle (this is Theorem 1 with $n = 3$). But as soon as one wants to do both trisecting and "quintisecting" (or "multisecting") then the classical Archimedean spiral is *the* right tool in the world of n-spiral functions. The following corollary generalizes these observations.

COROLLARY 4. *Let a and b be two positive integers such that $\log a/\log b$ is irrational. If F is a continuous or monotonic n-spiral function for both $n = a$ and $n = b$, then F is the Archimedean spiral.*

Proof. The irrationality of $\log a/\log b$ implies that the set $\{a^n b^m \mid n, m \in \mathbb{Z}\}$ is dense in $\mathbb{R}_+$ and the same argument as above applies. □

3. The Quadratrix of Hippias

Given a circle with center $(0, 0)$ and radius R, Hippias wanted to find a function f for which (see FIGURE 2)

$$\frac{f(a)}{f(b)} = \frac{\alpha}{\beta}, \quad \text{where } \tan\alpha = \frac{f(a)}{a} \quad \text{and} \quad \tan\beta = \frac{f(b)}{b}. \tag{3}$$

The reason for introducing such a curve (function) was geometric: using (3) it is possible to produce any rational proportion of angles by taking the same rational proportion of lengths along the vertical $\overline{OR}$ (the latter task could be performed using euclidean tools). We show below that such functions exist.

To be precise, we look for a continuous strictly decreasing function $f\colon [0, K] \to [0, R]$ such that $f(K) = 0$, $f(0) = R$ and, for all a, b in $[0, K)$, equation (3) holds, that is,

$$\frac{f(a)}{f(b)} = \frac{\arctan(f(a)/a)}{\arctan(f(b)/b)}. \tag{4}$$

(We use the principal value of the arctangent, i.e., $-\pi/2 < \arctan(x) < \pi/2$.)

The continuity of f at 0 (from the right) implies that $\lim_{a \to 0^+} f(a) = R$. As a consequence,

$$\lim_{a \to 0^+} \arctan\left(\frac{f(a)}{a}\right) = \frac{\pi}{2},$$

since $f(a)/a$ tends to $+\infty$ and the principal value of $\arctan t$ tends to $\pi/2$ as t tends to $+\infty$. This agrees nicely with the meaning of α in equation (3), and in FIGURE 2. So, if in equation (4) a tends to 0, we obtain

$$f(b) = \frac{2R}{\pi}\arctan\left(\frac{f(b)}{b}\right). \tag{5}$$

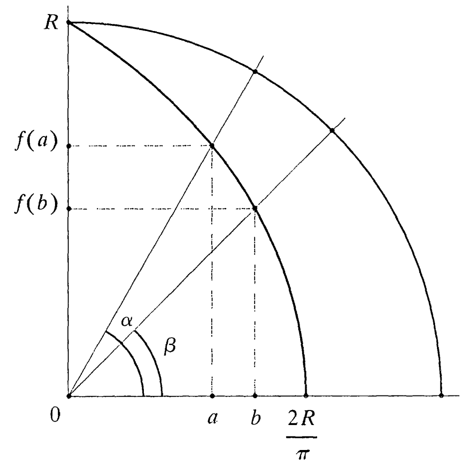

FIGURE 2
The quadratrix of Hippias.

It will be easier to determine the inverse function f^{-1}: $[0, R] \to [0, K]$ of f. If an f satisfying the above conditions exists, then its inverse function f^{-1} exists, is continuous and strictly decreasing, and satisfies $f^{-1}(0) = K, f^{-1}(R) = 0$, and (from (5) with $y = f(b)$),

$$y = \frac{2R}{\pi} \arctan\left(\frac{y}{f^{-1}(y)}\right).$$

This determines f^{-1} explicitly on $(0, R]$:

$$f^{-1}(y) = y \cot\left(\frac{\pi y}{2R}\right),$$

which implies that $f^{-1}(R) = 0$. By definition, $f^{-1}(0) = K$. If f^{-1} is continuous at 0, then

$$\begin{aligned} K = f^{-1}(0) &= \lim_{y \to 0^+} f^{-1}(y) = \lim_{y \to 0^+} y \cot\left(\frac{\pi y}{2R}\right) \\ &= \frac{2R}{\pi} \lim_{y \to 0^+} \cos\left(\frac{\pi y}{2R}\right) \frac{\pi y/(2R)}{\sin(\pi y/(2R))} = \frac{2R}{\pi}. \end{aligned}$$

(The last equality holds because $\lim_{t \to 0} t/\sin(t) = 1$.) So we see that f^{-1} is continuous at 0 (and hence f is continuous at K), only if $K = 2R/\pi$.

Conversely, $f^{-1}: [0, R] \to [0, K]$, defined by

$$f^{-1}(y) = \begin{cases} y \cot\left(\frac{\pi y}{2R}\right) & \text{if } y \in (0, R] \\ \frac{2R}{\pi} & \text{if } y = 0 \end{cases} \tag{6}$$

is continuous, at 0 as we have just seen and on $(0, R]$ because the cotangent function is continuous on $(0, \pi/2]$. Moreover, we have $f^{-1}(0) = 2R/\pi = K$, $f^{-1}(R) = 0$, and, as substitution shows,

$$\frac{x}{y} = \frac{\arctan(x/f^{-1}(x))}{\arctan(y/f^{-1}(y))}$$

for all $x, y \in (0, R]$. But this equation is clearly equivalent to (4). The only thing we still have to prove is that f^{-1}, and thus f, is strictly decreasing. Since f^{-1} is differentiable on $(0, R]$, we calculate

$$(f^{-1})'(y) = \cot\left(\frac{\pi y}{2R}\right) - \frac{\pi}{2R} \frac{y}{\sin^2(\pi y/2R)};$$

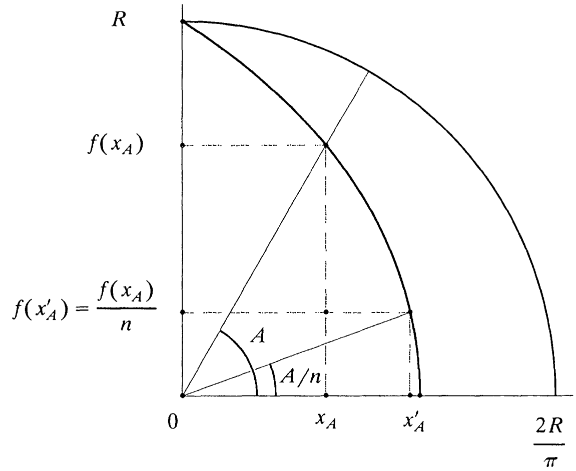

FIGURE 3
n-secting an angle with the quadratrix of Hippias.

this is negative on $(0, R]$ because

$$\cos\left(\frac{\pi y}{2R}\right)\sin\left(\frac{\pi y}{2R}\right) = \frac{1}{2}\sin\left(\frac{\pi y}{R}\right) < \frac{\pi y}{2R}.$$

So we have constructed the required f^{-1}, and thus also f.

This function can solve the *quadratura circuli*. However, we will use the quadratrix of Hippias not for squaring the circle but for trisecting the angle, a problem that had already been studied by Dinostratus, who lived nearly a century before Hippias.

The function f multisects any angle (see FIGURE 3): Given an angle A in the interval $\left(0, \frac{\pi}{2}\right)$, consider x_A in $\left[0, \frac{2R}{\pi}\right]$ such that $\tan A = f(x_A)/x_A$. If we divide the segment joining $(x_A, 0)$ with $(x_A, f(x_A))$ into n equal pieces, then the horizontal line $y = \frac{f(x_A)}{n}$ intersects $y = f(x)$ at a point $(x'_A, f(x'_A))$ with $f(x'_A) = f(x_A)/n$ and the line $y = \frac{f(x'_A)}{x'_A}x$ forms an angle of A/n radians with the x-axis, according to (3). Thus we see that to n-sect angles, we need only the particular case $\beta = \alpha/n$ of equation (3), that is, $f(b) = f(a)/n$ or $b = f^{-1}(f(a)/n)$. This motivates the following definition:

DEFINITION. *Given $n > 1$, an n-quadratrix is a bijective function $f\colon [0, K] \to [0, R]$ such that $f(0) = R$, $f(K) = 0$, and*

$$\frac{1}{n}\arctan\left(\frac{f(x)}{x}\right) = \arctan\left(\frac{f(x)/n}{f^{-1}(f(x)/n)}\right). \tag{7}$$

Note that equation (7) is the particular case $a = x$, $b = f^{-1}(f(x)/n)$ of (4). Following is a necessary and sufficient condition on n-quadratrix functions.

THEOREM 5. *A bijective function $f\colon [0, K] \to [0, R]$ is an n-quadratrix function if and only if there exists a function φ that satisfies equation (E_n), $\varphi(R) = \pi/2$, and*

$$f^{-1}(y) = \begin{cases} y\cot\varphi(y) & \text{if } y \neq 0, \\ K & \text{if } y = 0 \end{cases}.$$

Proof. With $f(x) = y \in [0, R)$, equation (7) becomes

$$\frac{1}{n}\arctan\left(\frac{y}{f^{-1}(y)}\right) = \arctan\left(\frac{y/n}{f^{-1}(y/n)}\right).$$

In other words, the function $\varphi(y) = \arctan(y/f^{-1}(y))$ must satisfy the functional equation (E_n): $\varphi(y/n) = \varphi(y)/n$. Thus $f^{-1}(y) = y\cot(\varphi(y))$ for $y \in [0, R)$, while $f(K) = 0$ means that $f^{-1}(0) = K$. □

Note that, when $\varphi(y) = \pi y/2R$, we obtain the quadratrix of Hippias and, as in Corollary 3, if a function, continuous or monotonic on $[0, K]$, is at the same time both a 3- and a 5-quadratrix (or a 2- and a 3-quadratrix, as mentioned in Corollary 4), then the function is the quadratrix of Hippias. Thus a situation similar to that of the Archimedean spiral applies to the classical quadratrix.

4. The Conchoid of Nicomedes

Given a positive constant $M > 0$, the point $O = (0,0)$ and the horizontal line $y = a$ with $a > 0$, the *conchoid of Nicomedes* is defined as the set of points $P = (x, y)$ in the plane such that $y > a$ and the distance from P to the intersection of the lines OP and $y = a$ is M (see FIGURE 4). Since the point of intersection has coordinates $(ax/y, a)$, we must have

$$\left(x - \frac{ax}{y}\right)^2 + (y-a)^2 = M^2;$$

this yields the cartesian equation $(y-a)^2(x^2+y^2) = M^2y^2$.

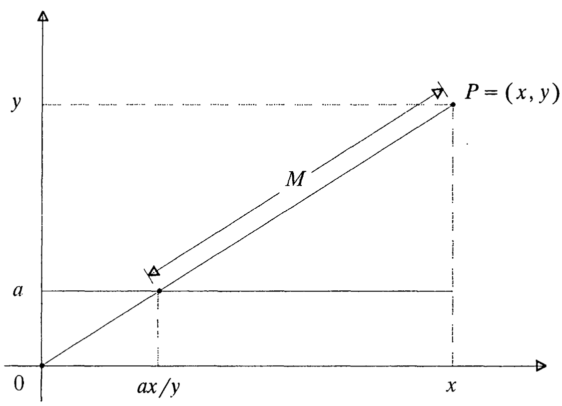

FIGURE 4
A generic point in a conchoid of Nicomedes.

Nicomedes discovered that, given an angle A, say in $\left(0, \frac{\pi}{2}\right)$, one can consider *the associated conchoid* to A with $a = 1$, $M = 2K$, and $K = 1/\cos A$ (see FIGURE 5) and use this to trisect the given angle A: The angle A is located in the first quadrant, opening out from the y-axis, K is determined, then the conchoid is drawn and the line BC determines the point C such that OC forms an angle of $A/3$ with the y-axis.

It is easy to see why this works. Since $\overline{OB} = K$ and $\overline{BP} = \overline{DC} = 2K$, we have $\overline{BC} = \overline{DC} \cos\alpha = 2K\cos\alpha$. Applying the law of sines to the triangle OBC we obtain $\overline{OB}/\sin\alpha = \overline{BC}/\sin\beta$. Thus

$$\sin\beta = \overline{BC}\sin\alpha/\overline{OB} = 2K\cos\alpha\sin\alpha/K = 2\sin\alpha\cos\alpha = \sin(2\alpha),$$

whence $\beta = 2\alpha$ and therefore $\alpha = A/3$.

One observes in this case that, for each angle, the trisection is obtained using the conchoid associated to that angle. Can we find a single function f—maybe after replacing the constant 1 or a by another function g—so that, by a similar construction, we can trisect *any* angle? To answer this question we introduce the following definition.

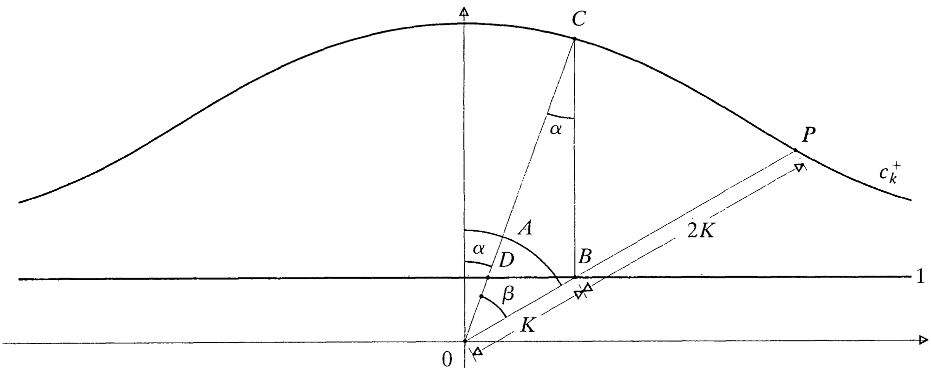

FIGURE 5
The conchoid of Nicomedes.

DEFINITION. *Two continuous functions f, g: $(0, +\infty) \to (0, +\infty)$ form a trisecting couple if $f(x)/x$ is a bijection from $(0, +\infty)$ onto $(\sqrt{3}, +\infty)$, $g(x)/x$ is a bijection from $(0, +\infty)$ onto itself, and, for every A in $(0, \frac{\pi}{2})$, there exists a unique point $x_A > 0$ such that the line $y = \frac{f(x_A)}{x_A}x$ makes an angle of $A/3$ (in radians) with the y-axis, and the line $y = \frac{g(x_A)}{x_A}x$ makes an angle A with the y-axis.* (*See* FIGURE 6.)

The assumption that $f(x)/x$ maps $(0, +\infty)$ onto $(\sqrt{3}, +\infty)$ is natural because, for all A in the interval $(0, \pi/2)$, we want to have a point x_A such that

$$\frac{f(x_A)}{x_A} = \cot\left(\frac{A}{3}\right) > \cot\left(\frac{\pi}{6}\right) = \sqrt{3}.$$

Similarly, $g(x)/x$ should attain all values in $(0, \infty)$, as does $\cot A$.

We proceed to characterize trisecting couples. If (f, g) is such a couple, then consider the (well defined) function ψ: $(0, \frac{\pi}{2}) \to (0, +\infty)$, defined by $\psi(A) = x_A$. By definition, one has the relations

$$\frac{g(\psi(A))}{\psi(A)} = \cot A \quad \text{and} \quad \frac{f(\psi(A))}{\psi(A)} = \cot\left(\frac{A}{3}\right). \tag{8}$$

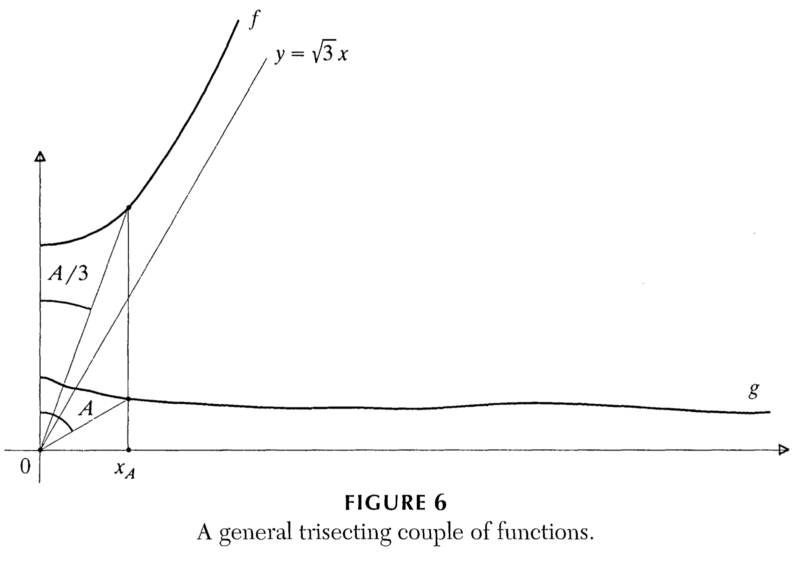

FIGURE 6
A general trisecting couple of functions.

From (8) and the bijectivity and continuity of $g(x)/x$ from $(0, +\infty)$ onto $(0, +\infty)$, it is immediate that ψ must also be bijective and continuous. Thus ψ^{-1} exists and (8) yields

$$g(x) = x\cot(\psi^{-1}(x)) \quad \text{and} \quad f(x) = x\cot\left(\frac{\psi^{-1}(x)}{3}\right). \tag{9}$$

It is easy to check that if ψ^{-1} is an arbitrary continuous bijection from $(0, +\infty)$ onto $(0, \pi/2)$, then the functions f and g given by (9) constitute a trisecting couple. We have proved the following theorem.

THEOREM 6. *A pair of functions* (f, g) *is a trisecting couple if and only if there exists a continuous bijection* ψ^{-1} *from* $(0, +\infty)$ *onto* $(0, \pi/2)$ *such that* (9) *holds.*

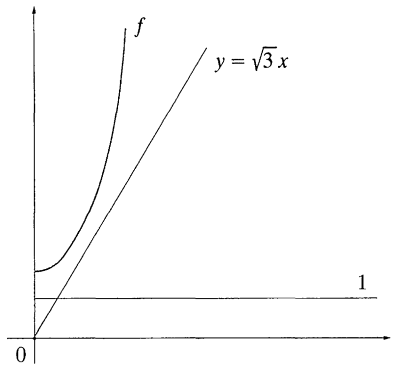

FIGURE 7
An example of a trisecting couple.

For example, when $g \equiv 1$ we get $f(x) = x\cot(\frac{\arctan x}{3})$; see FIGURE 7.

Thus there are interesting collections of trisecting couples. Other (non-Greek) trisecting curves, such as the trisectrix of Catalan or Tchirnhausen's cubic, Ceva's cycloid, and others, may be found in the literature (see, e.g., **[5, 9, 10, 12]**).

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NOTES

Confusing Clocks

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This note had its genesis in the early 1980s, when the third author was teaching at a community college in California and was asked a version of the following question by a student:

> Given a standard analog (two-hand) clock, are there times when the two hands could be interchanged to obtain another valid time (besides the obvious times when the two hands are at the same position)?

It was not hard to work out an answer to the question (see below), but the problem suggests many similar, and harder, questions. The question sat for years until the first author suggested it to one of his undergraduate students, the second author.

The most obvious of these questions regards a three-hand clock: Given a standard three-hand clock (with hour, minute, and second hands), are there times when the hands could be permuted in some way to obtain another valid time? Again, overlapping hands provide trivial solutions. We examine this question in the second section; in the last section we consider imperfect clocks.

The two-hand problem also appears in [**1**, **2**, **3**, **4**], with [**3**] giving the solution we give, [**2**, **4**] giving algebraic solutions, and [**1**] giving hints towards the solution below.

The two-hand clock Let us examine first the case of a two-hand, perfectly accurate, twelve-hour clock. As an example, take the time 2:00, when the hour hand points at 2 o'clock and the minute hand points at the 12 o'clock position. Permuting the hands, we do not get a valid clock position, since the hour hand pointing directly at 12 forces the minute hand to also point to 12. We notice that the position of the hour hand determines the position of the minute hand, so we can write the minute hand position as a function of the hour hand position. If we use the 60-minute scale on the clock face (so we measure the position of each hand as a real number in the interval $[0, 60)$—this is the usual scale for the minute hand, but not for the hour hand), and use h to represent the hour hand and $m(h)$ the minute hand, we have:

$$m(h) = 12h - 60\lfloor h/5 \rfloor \equiv 12h \pmod{60},$$

where $\lfloor x \rfloor$ means the greatest integer less than or equal to x. We will often indicate hand positions as ordered pairs (x, y), where x is the position of the hour hand and y the minute hand.

To answer the original question, we must find values of h for which $(m(h), h)$ is a valid clock position, i.e., for which $h = m(m(h))$. The last equation can be solved algebraically; this was done in [4], which includes an exhaustive list of all solutions. However, there is a nice way to "see" the answer (which appears in [3]): The point (a,b) in the plane represents one of the hand positions we are looking for if and only if (a, b) and (b, a) are both on the graph of the function $m(h)$, which has $[0, 60)$ for its domain and whose graph is shown in FIGURE 1.

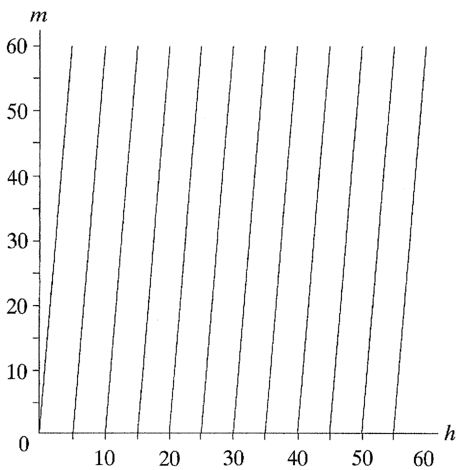

FIGURE 1
The graph of $m(h)$.

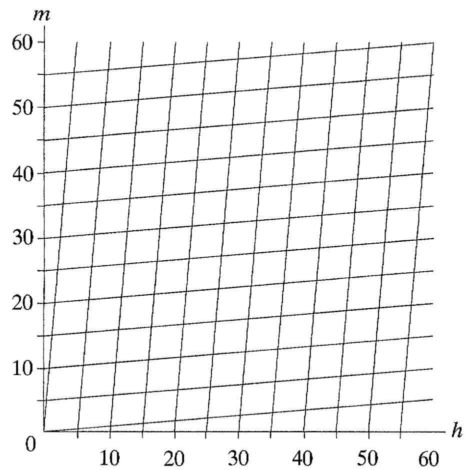

FIGURE 2
$m(h)$ and its reflection about the line $m = h$.

The point (b, a) is on this graph if and only if (a, b) is on the graph $(m(h), h)$, which is the reflection about the line $m = h$ of the graph above. Overlaying the two graphs, we have the graph shown in FIGURE 2, and the intersections are precisely the points we are looking for. There are 143 of them (the apparent intersection at $(60, 60)$ is the same as the one at $(0, 0)$); as mentioned above, they are catalogued in [4]. The 11 intersections that lie on the line $m = h$ are the trivial solutions where the hands are at the same position, so there are 132 non-trivial solutions.

Before discussing the three-hand clock, we mention one property of the greatest integer function that we use frequently: If r is a real number and a an integer, then $\lfloor r + a \rfloor = \lfloor r \rfloor + a$.

Three-hand clocks Next we consider a perfectly accurate three-hand clock. Our method of graphical intersections may not be so useful in the three-hand case. In this case, the lines representing the time are in three dimensions, and non-parallel lines may not intersect.

As with the minute hand, the position h of the hour hand determines the position of the second hand on our clock, via the function

$$s(h) = 720h - 60\lfloor 12h \rfloor \equiv 720h \pmod{60}.$$

With the additional hand on the clock come additional possible permutations of the hands. We represent hand positions as ordered triples (x, y, z), with z the position of the second hand. There are six possible permutations of the hands:

$$(1)\ (h, m(h), s(h)) \qquad (2)\ (m(h), h, s(h)) \qquad (3)\ (h, s(h), m(h))$$
$$(4)\ (s(h), m(h), h) \qquad (5)\ (s(h), h, m(h)) \qquad (6)\ (m(h), s(h), h)$$

The first is the normal position of the hands and always represents a valid time. The only obvious value for h that gives a valid time for any of the other permutations is the trivial one $h = 0$, or 12:00:00. Are there others?

Two-hand switches. The second permutation is just the hour hand–minute hand switch discussed above (i.e., $h = m(m(h))$), with the additional requirement that the two hands have the second hand in the same position: $s(h) = s(m(h))$. We examine the tables in [4] and see that there are no such solutions other than those 11 times when $h = m(h)$; that is, the hour and minute hands overlap.

The third permutation leaves the hour hand alone, and since its position determines the other two, the only solutions here are when $s(h) = m(h)$; i.e., once again, when the hands overlap. These are not particularly interesting cases. It is not hard to find them algebraically; we simply note that the second hand crosses the minute hand almost once a minute (59 times every hour, to be precise), so there are $12 \cdot 59 = 708$ such overlaps per rotation of the hour hand.

The fourth permutation (and the last two-hand switch to consider) is harder, though one might notice that mathematically this is the same as the second permutation, with the roles of $s(h)$ and $m(h)$ reversed. We must simultaneously solve $h = s(s(h))$ and $m(h) = m(s(h))$. Here, however, there are $720^2 - 1 = 518399$ solutions to the first equation; compiling a table and looking in it for solutions to the second didn't sound like much fun. Note that the second hand crosses the hour hand $12 \cdot 60 - 1 = 719$ times in one revolution of the hour hand, so there are at least that many solutions.

We solve $m(h) = m(s(h))$ first. For convenience, let $\alpha = h/5$ (α ranges from 0 to 12). We have, from routine calculations,

$$\begin{aligned} m(h) = m(s(h)) &\Leftrightarrow 12h - 60\lfloor h/5 \rfloor = 12(720h - 60\lfloor 12h \rfloor) \\ &\qquad - 60\lfloor (720h - 60\lfloor 12h \rfloor)/5 \rfloor \\ &\Leftrightarrow \lfloor 144h - \lfloor h/5 \rfloor \rfloor = 719h/5 \\ &\Leftrightarrow \lfloor 720\alpha - \lfloor \alpha \rfloor \rfloor = 719\alpha. \end{aligned}$$

Clearly, 719α must be an integer for this last equation to be true; conversely, if 719α is an integer, then $\lfloor 720\alpha - \lfloor \alpha \rfloor \rfloor = \lfloor 719\alpha + \alpha \rfloor - \lfloor \alpha \rfloor = 719\alpha + \lfloor \alpha \rfloor - \lfloor \alpha \rfloor = 719\alpha$. Thus the last equation holds if and only if 719α is an integer.

So we need α in $[0, 12)$ such that 719α is an integer. As 719 is prime, α must be of the form $n/719$ for some integer n satisfying $0 \le n < 12 \cdot 719 = 8628$. Now we check which of these 8628 solutions to $m(h) = m(s(h))$ also satisfies $h = s(s(h))$. Let $\alpha = \frac{n}{719}$ $(0 \le n \le 8627)$ be such a solution. With some manipulation, the requirement $h = s(s(h))$ reduces to $60\lfloor 12 \cdot 720h \rfloor = (720^2 - 1)h$. Substituting 5α for h in this equation gives

$$60\left\lfloor 12 \cdot 720 \frac{5n}{719} \right\rfloor = (720^2 - 1)\frac{5n}{719} \Leftrightarrow \left\lfloor 12 \cdot 720 \frac{5n}{719} \right\rfloor = \frac{721n}{12}.$$

The right side of this last equation is an integer only if $12|n$. There are 719 multiples of 12 in the interval $[0, 8628)$; as we already know, there are at least this many solutions; this is all of them. Once again, there are no non-trivial solutions.

This hand-switch can also be approached using *Mathematica* to do some of the arithmetic, but *Mathematica* doesn't save as much work on this permutation as it does on the cyclic permutation below.

Three-hand cyclic permutations. The two permutations we haven't considered yet are those that move all three hands. It will turn out that we need to solve only one of these; the other will be a consequence. Consider the fifth permutation above. If $(h, m(h), s(h))$ represents a valid time, then the requirements for $(s(h), h, m(h))$ to also represent a valid time are $h = m(s(h))$ and $m(h) = s(s(h))$.

We were getting tired of doing mod 60 algebra by hand at this point, so we asked *Mathematica* for help. *Mathematica* was not a big fan of non-prime moduli for modular arithmetic either, but we finally settled on the `Reduce` command, handling the modulus ourselves. This command simplifies equations, attempting to solve for the variable(s) we specify (h in the example below); the equations generated by `Reduce` are equivalent to the original equations and contain all possible solutions. If we define $m(h) = 12h$ and $s(h) = 720h$, then the command

```
In[1] := Reduce[m[s[h]]-60k == h && m[h] == s[s[h]]-60 j, h]

Out[1]= 8639 j=518388 k && h=(60 k)/8639
```

does the trick: Keeping in mind that j and k must be integers, and checking (again with *Mathematica*) that 518388 and 8639 are relatively prime, we find that $518388|j$ and $8639|k$. Since $8639|k$, the equation $h = 60k/8639$ reduces modulo 60 to $h \equiv 0$. In other words, the only solution is the trivial one, where all three hands overlap at 12:00:00.

Why don't we have to consider the remaining permutation? Let's say we had a solution for the sixth permutation $(m(h), s(h), h)$; that is, some value h_0 such that $s(h_0) = m(m(h_0))$ and $h_0 = s(m(h_0))$. We claim that then h_0 is a solution for the permutation we considered above! We must show that $h_0 = m(s(h_0))$ and $m(h_0) = s(s(h_0))$. The first calculation is as follows:

$$h_0 = s(m(h_0)) \equiv 720(m(h_0)) \equiv 720 \cdot 12h_0 \equiv m(s(h_0)) \pmod{60}.$$

Since h_0 and $m(s(h_0))$ are both non-negative and strictly less than 60, we conclude that $h_0 = m(s(h_0))$. (In fact, we just showed that $s(m(h)) = m(s(h))$ for any h!) Now we have

$$s(s(h_0)) = s(m(m(h_0))) = m(m(s(h_0))) = m(h_0);$$

this takes care of the last permutation.

We have shown that for a perfect three-hand clock, there are no times when hands can be interchanged to obtain valid clock positions, except for the obvious ones when the hands overlap. In other words, if you've a sharp eye, you can always tell what time it is on such a clock, even if the hands are installed in some permuted order.

Is your clock perfect? Ours aren't. How strict must manufacturing tolerances be to ensure that there are no nontrivial permutations of the hands that give valid times? Let's investigate.

We'll assume that the spindles that turn the hands are geared together accurately, so we're not worried about the relative speeds of the hands. Our concern is with the proper alignment of the hands. Notice that there are 11 times in a 12-hour period at which the minute hand and hour hand will overlap. If we take one of these and turn the clock face so that the minute and hour hands are pointing at 0 (i.e., 12 o'clock), we see that any strange mounting of the hands can be considered as a mis-mounting of the second hand only, with the minute and hour hands mounted perfectly.

With these assumptions, we have the following function for the second hand in terms of the hour hand:

$$\hat{s}(h) = 720h + o - 60\lfloor 12h + o/60 \rfloor \equiv 720h + o \pmod{60},$$

where o stands for "offset." The function that describes the position of the minute hand is still $m(h) = 12h - 60\lfloor h/5 \rfloor$. We want to find the minimal value of o that gives a non-trivial solution for one of the five non-trivial permutations in the last section.

It is immediate from the work done for the perfect clock that permutations 2 and 3 give no non-trivial solutions for any values of o. Permutation 4 is again more work, but the same argument as for the perfect clock again does the trick, with o added to the appropriate places in the calculations (that is, with $\hat{s}$ replacing s).

Finally, we have to consider the two permutations that move all three hands. We claim again that we need only consider one of them. Assume h_0 is a value such that $(m(h_0), \hat{s}(h_0), h_0)$ is a valid time, that is,

$$h_0 = \hat{s}(m(h_0)) \quad \text{and} \quad \hat{s}(h_0) = m(m(h_0)).$$

It is not true now that $(\hat{s}(h_0), h_0, m(h_0))$ is a valid time, but we claim that $(\hat{s}(12h_0), 12h_0, m(12h_0))$ is valid, so $12h_0$ gives the desired solution for the other permutation (with $12h_0$ reduced modulo 60, if necessary). We must show that $12h_0 = m(\hat{s}(12h_0))$ and $m(12h_0) = \hat{s}(\hat{s}(12h_0))$. Here are the calculations:

$$12h_0 = 12\hat{s}(m(h_0)) \equiv m(m(m(m(h_0)))) \equiv m(\hat{s}(12h_0)) \pmod{60};$$
$$m(12h_0) = m(m(h_0)) = \hat{s}(h_0) = \hat{s}(\hat{s}(m(h_0))) = \hat{s}(\hat{s}(12h_0)).$$

So we consider only the permutation $(m(h), \hat{s}(h), h)$. We want to find the smallest $|o|$ that will give a non-trivial hand-switch. We proceed with the help of *Mathematica* as for the perfect clock (s and m are the same functions as before: $s(h) = 720h$, $m(h) = 12h$).

```
In[1] := Reduce[s[m[h]] + o- 60 m == h && m[m[h]] == s[h] + o- 60
j,{h, o}]
                 -60 (j-m)                60 (8639 j-576 m)
Out[1]= h ==------------&& o ==-----------------
                     8063                        8063
```

Working with this output, we find that 8639 and 576 are relatively prime; therefore there are integers m and j such that $8639j - 576m = 1$. As this will give us the smallest value for $|o|$, we find them (using the Euclidean algorithm or *Mathematica*) and get $m = 8624$, $j = 575$. Substituting these into $h = (60(-j + m))/8063$, we obtain $h = (60 \cdot 8049)/8063$, with the offset $o = 60/8063$. This translates into a time of approximately 11:58:44.9981, with hand positions

$$\left(\frac{482940}{8063}, \frac{473700}{8063}, \frac{362820}{8063}\right).$$

The hands don't overlap here, but we get another valid hand position by permuting the hands:

$$\left(\frac{473700}{8063}, \frac{362820}{8063}, \frac{482940}{8063}\right).$$

It is easy to check that this last position is indeed valid. We get a similar solution for $o = -60/8063$.

So, unless you can be certain that your second hand is mounted no more than 60/8063 seconds (about 8 thousandths of a second) off of vertical (at noon), you'd better be sure you know which hand is which!

Acknowledgment. We thank the referee for pointing out Steinhaus's book [**3**] and for improving the exposition.

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The Steady State Sabbatical Rate

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Introduction How many faculty at a research university are likely to be on sabbatical at any given time? Well, we are each eligible for one every seven years, so about one seventh, or 14%, right? Obviously not. Some of us do not survive until the seventh year to become eligible, some do not choose to apply for leave, and, finally, some applications are rejected. So it must be lower. What do you think it is? 10%? 7%? 3%? With all these effects going on, it seems that we cannot hope to predict the answer. And yet the solution depends upon only two modest assumptions and knowledge of elementary linear algebra. Thus, this problem can be used in various undergraduate classes to illustrate the use of matrices and eigenvalues in the real world. It may even hold some special appeal for faculty and administrators who'd like to be able to predict these things. For example, at my university it was suggested that a 3% cap be accepted on the sabbatical rate: in a single year, no more than 3% of the faculty would be on sabbatical. This cap would assure that an unusually large number of requests do not pile up in a single year, leaving the university sorely understaffed. But would such a cap merely "balance out the waves" of irregular demand, or would it intrinsically change the frequency of sabbaticals in the long term?

Background: Perron-Frobenius theory In addition to standard elementary linear algebra, we shall need some conclusions from the Perron-Frobenius Theorem (see, e.g., Berman and Plemmons [**1**, pp. 26–31] or Gantmacher [**2**]). For the sake of completeness, we now prove the results we shall need. A square matrix A is called *irreducible* if there is no permutation matrix P such that $PAP^t = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$ with square blocks A_{11} and A_{22} on the diagonal. A matrix induces a directed graph when we place an arc from i to j whenever the entry $a_{i,j}$ is nonzero. An irreducible matrix always induces a digraph having a directed path from each vertex i to every other vertex j. This is called *strongly connected*. In this case, the matrix $(I+A)^{n-1}$

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will have strictly positive entries because the required path between each pair of vertices can have length at most $n-1$. We let v_i denote the ith coordinate of $\mathbf{v}$.

PERRON-FROBENIUS THEOREM. *If A is irreducible and nonnegative, then A must have an eigenvalue λ_1, of maximum modulus, that is real and positive. Moreover, the multiplicity of λ_1 must be one, and λ_1 has an eigenvector $\mathbf{y}$ whose coordinates are all positive.*

Proof. For each nonnegative vector $\mathbf{x}$ define $r(\mathbf{x})$ to be the largest positive constant for which $r(\mathbf{x})\mathbf{x} \leq A\mathbf{x}$. If ρ is the maximum of $r(\mathbf{x})$ over all nonnegative $\mathbf{x}$, then for some $\mathbf{y}$, we have $\rho\mathbf{y} \leq A\mathbf{y}$. We wish to show that no coordinate j gives the strict inequality. If one did, we'd have $A\mathbf{y}-\rho\mathbf{y} \geq 0$, but $A\mathbf{y}-\rho\mathbf{y} \neq 0$. Let $\mathbf{z}=(I+A)^{n-1}\mathbf{y}$. Then $\mathbf{z}$ is strictly positive, and we find that

$$A\mathbf{z}-\rho\mathbf{z}=A(I+A)^{n-1}\mathbf{y}-\rho(I+A)^{n-1}\mathbf{y}=(I+A)^{n-1}(A\mathbf{y}-\rho\mathbf{y}).$$

Since $A\mathbf{y}-\rho\mathbf{y}>0$ and $(I+A)^{n-1}$ is strictly positive, we conclude that $A\mathbf{z}-\rho\mathbf{z}$ is also strictly positive. Thus, $r(\mathbf{z})>\rho$ contradicting the definition of ρ. Therefore, we must have $A\mathbf{y}=\rho\mathbf{y}$, so ρ is an eigenvalue with nonnegative eigenvector $\mathbf{y}$. If any coordinate of $\mathbf{y}$ is 0, irreducibility of A assures that there is a zero coordinate $y_i=0$ and a nonzero coordinate $y_j>0$, with a corresponding nonzero entry $a_{i,j}>0$. But now we notice that $(A\mathbf{y})_i>0=\rho y_i$. Since this contradicts $A\mathbf{y}=\rho\mathbf{y}$, we conclude that $\mathbf{y}$ must be strictly positive.

If the multiplicity of $\rho=\lambda_1$ exceeds one, select a second eigenvector $\mathbf{z}$. Now for $\varepsilon=0$, the vector $\mathbf{w}=\mathbf{y}-\varepsilon\mathbf{z}$ is positive. Find the largest number ε for which $\mathbf{w}$ is nonnegative, then $\mathbf{w}$ is a nonnegative eigenvector with some zero coordinates. If $\mathbf{w}$ also has a nonzero coordinate, we shall locate a zero coordinate $w_i=0$ and a nonzero coordinate $w_j>0$ with a corresponding nonzero entry $a_{i,j}>0$, leading to the contradiction $(Aw)_i>0=\rho w_i$. Therefore $\mathbf{w}$ must be identically $\mathbf{0}$. That is, $\varepsilon\mathbf{z}=\mathbf{y}$, so the multiplicity of ρ must be one.

Finally, let λ, real or complex, be any other eigenvalue for A. Then $A\mathbf{v}=\lambda\mathbf{v}$. Define a new nonnegative vector $\mathbf{w}$ by letting $w_i=|v_i|$. Taking absolute values in each coordinate of the eigenvector equation and using the triangle inequality gives

$$|\lambda|w_i=|\lambda|\,|v_i|=\left|\sum_{j=1}^{n} a_{i,j}v_j\right| \leq \sum_{j=1}^{n} a_{i,j}|v_j| = \sum_{j=1}^{n} a_{i,j}w_j.$$

This says that $|\lambda|\mathbf{w} \leq A\mathbf{w}$, so $|\lambda| \leq r(\mathbf{w}) \leq \rho=\lambda_1$, as required.

The solution We can now return to the sabbatical problem. To start the analysis, we partition the faculty into seven classes, X_1 through X_7. For $i \leq 5$, X_i contains all faculty in their ith year of service or in their ith year following their last sabbatical. These people are not yet eligible for sabbatical. The class X_6 contains everyone who is eligible to apply for sabbatical. We will assume that X_6 contains *everyone* with six or more years of service since his or her sabbatical. (Many universities also require sabbatical candidates to be tenured; for simplicity, we assume that everyone with six years' service is eligible to apply. In practice, there are few exceptions to this assumption, and if there are any exceptions, they can be removed from the model entirely without affecting the solution.) Finally, class X_7 comprises all faculty currently on sabbatical.

We shall let $x_i(t)$ denote the fraction of the faculty in class X_i in year t. Now the fractions $x_i(t)$ vary from year to year; we call the distribution a *steady state* if every x_i remains constant from one year to the next. It is certainly not obvious that any steady state exists. We shall soon show, however, that not only does a steady state exist, but it is unique and stable in the sense that for any starting distribution of faculty among the seven classes, the proportions $x_i(t)$ approach the steady state over time.

Thus, our goal is to find the seven limits $\lim_{t\to\infty} x_i(t) = y_i$. We call these values y_i the "steady state" proportions of our faculty. In particular, y_7 is the "steady state sabbatical rate" of the title. But how can we find these steady state rates? Are we even justified to assume that these limits exist? After all, any human endeavor is filled with variability. Might not the sequence $x_7(t)$ vary from year to year without ever approaching a limit? Maybe $x_7(t)$ is another example of chaos.

In the real world the steady state is never attained. We cannot plan the lives of hundreds of people to reproduce the assumptions of a mathematical model. So why should we care about steady state values? Because they represent an unbiased, good faith estimate of the *average* number of people that ought to appear in each class over an extended period of time. The administration's request for a "cap" reflects the legitimate concern that a change in policy might artificially release pent-up demand for sabbaticals that vastly exceeds the steady state figure. It is reasonable to try to mute, or limit this transient effect. But if we accept a cap below the steady state sabbatical rate y_7, pent-up demand can only continue to grow more and more out of balance.

A steady state We will show that the steady state limits *do* exist, if we may make two assumptions. First, we assume that there is a steady retention rate, r: each year, a certain fraction r of the faculty who taught in the previous year return. The complementary figure $1-r$ is the turnover rate. All right, I admit this is not perfectly constant from year to year, but at my university it varies from about 89% to 95%. That will be constant enough. We shall assume, in particular, that the turnover rate hits all classes equally. This may seem counter-intuitive, but it is a reasonable simplifying assumption. For purists who refuse to accept this choice of convenience, we shall return to this issue later. Accepting a sabbatical often includes a promise to return for one or two years of service following the leave, so one might expect the retention rate for class X_7 to be 100%, but even here there will be terminations due to death, debilitating illness, and broken promises.

Our second assumption is that a given fraction of X_6, say p, will be granted sabbatical each year. This may seem hard to predict because of the variations of how many choose to apply, and how many of these are subsequently granted. In fact, we can view the proportion p actually to be the product of two other rates, $p = a \times s$, where a is the application rate, the fraction of those eligible who choose to apply, and s is the success rate, the fraction of applications granted in a typical year. While we may never "know" these values precisely, we don't even need to know them individually. It will suffice if we assume that their product p exists and is constant. If we don't know p exactly, we can still draw conclusions from reasonable estimates of p. At Western Michigan University, I observe that p seems to lie between 0.2 and 0.4. Such an estimate is adequate for our purposes.

Now it is easy to see that with r and p as above, the classes of our faculty satisfy a system of eight equations:

$$
\begin{aligned}
x_2(t+1) &= rx_1(t) & x_6(t+1) &= rx_5(t) + (1-p)rx_6(t)\\
x_3(t+1) &= rx_2(t) & x_7(t+1) &= prx_6(t)\\
x_4(t+1) &= rx_3(t) & x_1(t+1) &= 1-r+rx_7(t)\\
x_5(t+1) &= rx_4(t) & x_1(t)+x_2(t)&+x_3(t)+x_4(t)+x_5(t)+x_6(t)+x_7(t) = 1.
\end{aligned}
$$

The first four equations report retention from one year to the next. The fifth also adds those not given a sabbatical who remain in X_6. The sixth reports how many go on sabbatical, and the seventh counts new hires plus those returning from sabbatical. The final equation adds everyone, to get 100%. It is interesting to note that, in the steady state, when each $x_i(t+1) = x_i(t)$, the eighth equation is the sum of the first seven:

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = r(x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7) + 1 - r$$
$$\Rightarrow x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 1.$$

If we define the column vector $\mathbf{x}(t) = (x_1(t), x_2(t), x_3(t), x_4(t), x_5(t), x_6(t), x_7(t))^t$ and the transition matrix

$$A = \begin{pmatrix} 1-r & 1-r & 1-r & 1-r & 1-r & 1-r & 1 \\ r & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & r & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & r & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & r & r-rp & 0 \\ 0 & 0 & 0 & 0 & 0 & rp & 0 \end{pmatrix},$$

then the system is nicely presented as the matrix equation $\mathbf{x}(t+1) = A\mathbf{x}(t)$. The system approaches a steady state if and only if $A^n\mathbf{x}(t)$ approaches a limiting vector. Notice that A is irreducible, because the positive entries down the subdiagonal plus the entry $a_{1,7} = 1$ represent a directed seven-cycle among the seven classes. Since this makes the underlying digraph strongly connected, the matrix is irreducible. Now the Perron-Frobenius theorem assures the existence of a strictly positive eigenvector for the largest eigenvalue.

It is not immediately evident what this dominant eigenvalue is, but we might notice that the transpose A^t has $\mathbf{j} = (1, 1, 1, 1, 1, 1, 1)^t$ as an eigenvector for the eigenvalue 1. In other words, A^t is *stochastic*. Thus $\lambda_1 = 1$ for both A^t and A. To find an eigenvector for $\lambda_1 = 1$ in A, let $\mathbf{y} = (y_1, y_2, y_3, y_4, y_5, y_6, y_7)^t$ and assume that $A\mathbf{y} = \mathbf{y}$. If we set $y_1 = a$, we quickly find the eigenvector

$$\mathbf{y} = a\left(1, r, r^2, r^3, r^4, \frac{r^5}{1 - r + rp}, \frac{r^6 p}{1 - r + rp}\right)^t.$$

But $\mathbf{y}$ represents the fractions in each class, so we should have $\mathbf{y} \cdot \mathbf{j} = 1$. This implies that

$$a = \frac{(1-r)(1-r+rp)}{1 - r + rp - r^7 p}, \text{ so}$$

$$\mathbf{y} = \frac{(1-r)(1-r+rp)}{1 - r + rp - r^7 p}\left(1, r, r^2, r^3, r^4, \frac{r^5 p}{1 - r + rp}, \frac{r^6 p}{1 - r + rp}\right)^t.$$

In particular, the steady state sabbatical rate is $y_7 = (1-r)r^6 p/(1 - r + rp - r^7 p)$.

The Perron-Frobenius theorem has been very kind to us. Not only can we verify that $\mathbf{y}$ is a steady state dominant eigenvector, but we know that, up to scaling, $\mathbf{y}$ is unique. Thus, the $\mathbf{y}$ we have found is the only one whose components sum to 100%. The characteristic polynomial of A can be found easily via *Maple*, or by hand via expansion along the first row. Upon factoring, we get

$$\det(xI - A) = (x-1)(x^6 + rpx^5 + r^2px^4 + r^3px^3 + r^4px^2 + r^5px + r^6p)$$
$$= (x-1)q(x).$$

The degree 6 factor $q(x)$ has six complex roots; we shall now show that each has modulus smaller than r. First, substitute $x = rz$ and factor, to get

$$q(x) = r^6(z^6 + pz^5 + pz^4 + pz^3 + pz^2 + pz + p).$$

Upon multiplying by $(z-1)$ we obtain

$$(z-1)q(x) = r^6(z^7 - (1-p)z^6 - p).$$

Now any root must satisfy $z^7 = (1-p)z^6 + p$. If we assume that $|z| \geq 1$, the triangle inequality implies that

$$|z|^7 = |(1-p)z^6 + p| \leq (1-p)|z|^6 + p \leq (1-p)|z|^6 + p|z|^6 \leq |z|^6.$$

This certainly requires $|z| = 1$, and to get equality above, we need $z^6 = 1 = z^7$. Thus we have shown that the only root with $|z| \geq 1$ is $z = 1$. It follows that every root of $q(x)$ has $|z| < 1$, or $|x| < r$, as we had claimed.

No root is repeated because $q(x)$ and $q'(x)$ are relatively prime for every p in the open interval $0 < p < 1$. This can be checked using *Maple*, or by hand computation. It is actually more convenient to work with the degree seven polynomial $z^7 - (1-p)z^6 - p$. Applying the Euclidean algorithm to it and its derivative reduces to the real number $7^4 p^2 [6^6(1-p)^7 + 7^7 p]$. Since this expression has no root on the open interval $0 < p < 1$, we may conclude that the gcd reduces to 1, and so A has no repeated eigenvalues. Now we let $\mathbf{v}_i$ denote the six eigenvectors for these complex eigenvalues, and we observe that any nonnegative starting vector can be written as a linear combination

$$\mathbf{x}(0) = c_0\mathbf{y} + \sum_{i=1}^{6} c_i \mathbf{v}_i.$$

Now we see that

$$A^n\mathbf{x}(0) = \mathbf{x}(n) = c_0\mathbf{y} + \sum_{i=1}^{6} c_i \lambda_i^n \mathbf{v}_i.$$

Since each $|\lambda_i^n| < r^n$, it is clear that $A^n\mathbf{x}(0)$ approaches $c_0\mathbf{y}$ as n approaches ∞. Thus, the steady state is unique, and any starting input converges to the steady state. Since $\mathbf{y}\cdot\mathbf{j} = 1$, the constant c_0 equals 1.

In the real world What does all this mean in the real world? At Western Michigan University the retention rate seems to vary from year to year from a low of 89% to a high of 95%. If we assume that, on average, $r = 0.92$, then the steady state sabbatical rate is

$$y_7 = \frac{0.048\,508p}{0.08 + 0.362\,153p}.$$

Suppose that about a third of eligible professors will apply for sabbatical and that 75% of applications are granted. Then $p = 0.75 \times 0.33 = 0.25$, and $y_7 = 7.11\%$. If, instead, we let p range from 0.2 to 0.4, then y_7 increases from 6.36% to 8.63%. So it is reasonable to expect 6% to 9% of the faculty to be on sabbatical in any single year. What about that proposed cap of 3%? Artificially imposing a cap—any cap—redefines the transition matrix, creating a new steady state. For example, the 3% cap forces $y_7 = 0.03$, which, in turn, requires the proportion p to be given by the formula

$$p = \frac{y_7(1-r)}{r^6(1-r) + y_7 r(r^6-1)} = \frac{0.03(1-r)}{r^6(1-r) + 0.03r(r^6-1)}.$$

For $r = 92\%$, we find that $p = 6.38\%$, a shockingly low proportion. Think about it. If about 30% of those in class X_6 regularly apply, then we must have a success rate of only 21% to achieve p. Thus, a typical faculty member would need to apply about *five* times before a sabbatical is granted. That's one sabbatical every eleven years. Perhaps we should change the name to an *undecematical*. In contrast, a cap of 7% limits p to 24.18%, a figure large enough to permit an application rate of 33%, together with a success rate of 73%. This seems more in line with the original spirit of the sabbatical concept.

Another interesting observation is how large the class of those eligible (and waiting) for sabbatical becomes. With a 3% cap and $r = 92\%$, we find that x_6 approaches 51.15%. Over half the faculty is in line for sabbatical. With a 7% cap, the result is that x_6 approaches 31.46%.

The fine print When we first introduced the model, we brushed aside the issue of how reasonable it is to use the same retention rate for all classes. If you find the constant uniform retention implausible, simply replace the single rate r by a specific rate for each class r_i. The transition matrix becomes

$$A = \begin{pmatrix} 1-r_1 & 1-r_2 & 1-r_3 & 1-r_4 & 1-r_5 & 1-r_6 & 1 \\ r_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & r_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & r_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & r_5 & r_6 - r_6 p & 0 \\ 0 & 0 & 0 & 0 & 0 & r_6 p & 0 \end{pmatrix}.$$

It may seem strange that the rate r_7 appears nowhere in the matrix, but this is appropriate since everyone currently on sabbatical either returns to class X_1 or is replaced by a new hire, also in class X_1. In effect, 100% of class X_7 moves to class X_1 the following year. It remains true that $A^t\mathbf{j} = \mathbf{j}$. Now the eigenvector for $\lambda_1 = 1$ in A is

$$\mathbf{y} = a\left(1, r_1, r_1r_2, r_1r_2r_3, r_1r_2r_3r_4, \frac{r_1r_2r_3r_4r_5}{1 - r_6 + r_6 p}, \frac{r_1r_2r_3r_4r_5r_6 p}{1 - r_6 + r_6 p}\right).$$

Since $\mathbf{y}\cdot\mathbf{j} = 1$, we find

$$a = \frac{1 - r_6 + r_6 p}{(1 + r_1 + r_1r_2 + r_1r_2r_3 + r_1r_2r_3r_4)(1 - r_6 + r_6 p) + r_1r_2r_3r_4r_5(1 + r_6 p)}.$$

Thus, the steady state rate has become

$$y_7 = \frac{r_1r_2r_3r_4r_5r_6 p}{(1 + r_1 + r_1r_2 + r_1r_2r_3 + r_1r_2r_3r_4)(1 - r_6 + r_6 p) + r_1r_2r_3r_4r_5(1 + r_6 p)}.$$

This is not much different from the previous answer. In fact, if we set all the retention rates to a common value, the steady state sabbatical rate again reduces to

$$y_7 = \frac{(1-r)r^6 p}{1 - r + rp - r^7 p}.$$

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A Quadratic Residues Parlor Trick

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Statements Choose a prime $p > 3$ such that $p \equiv 3 \pmod 4$, but don't tell me what p is. Now do (or have your computer do) the following calculations: (1) determine which numbers between 0 and $p/2$ are quadratic residues modulo p (i.e., are congruent to squares, mod p); (2) find the sum of these "low" quadratic residues; (3) replace said sum by its least non-negative residue r (mod p). Here are three examples:

p	quadratic residues (mod p)	"low" q.r.'s ($<p/2$)	sum of low q.r.'s	r
7	$1, 2, 4$	$1, 2$	3	3
11	$1, 3, 4, 5, 9$	$1, 3, 4, 5$	13	2
19	$1, 4, 5, 6, 7, 9, 11, 16, 17$	$1, 4, 5, 6, 7, 9$	32	13

Now here's the parlor trick: If you tell me what r is, I'll tell you what p was. (Fairly quickly, without a computer.) Conversely, if you tell me p, I can quickly compute r without using any information about quadratic residues.

Challenge. Find the pattern before reading the next paragraph. You might start by computing and then plotting the points (p, r) for perhaps twenty more values of p. If this reveals the pattern to you, you'll then be able to go from p to r; but can you go in the opposite direction?

OK, we won't keep you in suspense. The relevant theorem is the following:

THEOREM 1. *Let* $P = \{p > 3\colon p \text{ is prime and } p \equiv 3 \pmod 4\}$. *For fixed* $p \in P$, *let* $r = R(p)$ *be the least non-negative residue (mod p) of the sum of those quadratic residues (mod p) that lie in the interval* $(0, p/2)$. *Then*:

(a) p *is the largest prime factor of* $16r + 1$. (*In particular,* $r \neq 0$.) *Moreover,* $p = (16r + 1)/m$ *where* $m = 3$, 7, 11, *or* 15, *and m is the smallest of these four values such that* $(16r + 1)/m$ *is prime.*

(b) *Conversely, if* r^* *is a positive integer such that* $(16r^* + 1)/m$ *is prime for some* $m \in \{3, 7, 11, 15\}$, *then* $r^* = R(p^*)$ *for some* $p^* \in P$.

Notice that Theorem 1(a) implies that *the function* $R\colon P \to \mathbb{Z}^+$ *is one-to-one*—different P-primes p yield different r's. (This was the assertion of part (a) of **[2]**.) Before proving the theorem, let's give some examples. Suppose you follow the instructions at the beginning (choose p, etc.) and then tell me that $r = 13$. Since $16r + 1 = 209 = 11 \cdot 19$, Theorem 1(a) then tells me that $p = 19$ (and $m = 11$), agreeing with the table above. (Even for a number larger than 209, the fact that there are only four possible m's makes the factoring problem fairly manageable.) However, suppose you decide to cheat: Without following instructions, you just pick a number at random and say, "r is 23." "Oh, no it isn't!", I can answer, having quickly computed $16 \cdot 23 + 1 = 369 = 3^2 \cdot 41$, which implies that $369/m$ is *not* prime for any of the four possible m's. (By a similar argument, the range of R contains no integer of the form $9n + 5$, which, in particular, solves part (b) of **[2]**. Can you find other such forms?)

Going in the opposite direction, if $p \in P$ is given then Theorem 1(a) implies $pm \equiv 1 \pmod{16}$, so that m is the multiplicative inverse of p (mod 16), and since

$0 < m < 16$ this determines m completely:

$$m \equiv 14 - p \pmod{16}; \quad 0 < m < 16. \tag{1}$$

Equivalently,

$$m = 16\lfloor (p+2)/16 \rfloor + 14 - p.$$

Since $r = (pm-1)/16$, it follows that we can express r explicitly in terms of p, without any calculation of quadratic residues. (*The points* (p, r) *lie on just four straight lines*, corresponding to the four values of p mod 16. Did you discover this already, in response to the "Challenge" above?) For example, if $p = 19$, then (1) gives $m = 11$ and hence $r = (19 \cdot 11 - 1)/16 = 13$, as we expect.

In proving Theorem 1, we will need the following related result:

THEOREM 2. *Let $p \in P$ and let S_p, S'_p be, respectively, the sums of those quadratic residues and quadratic nonresidues* (*mod* p) *that lie in* $(0, p/2)$. *Then* $S_p \equiv S'_p \pmod{p}$ *and* $(S_p - S'_p)/p \equiv (p+1)/4 \pmod{2}$.

Theorem 2 is known (we give an elementary proof below) and is of interest in its own right. But I haven't found Theorem 1 in print; references would be appreciated.

Proofs. First, some notation. As above, we let

$$P = \{ p > 3\colon p \text{ is prime and } p \equiv 3 \pmod{4} \}.$$

For any finite set A of numbers, $\#A$ and $s(A)$ will denote, respectively, the number of elements in A and the sum of those elements. Also, for $p \in P$, let $L_p = \{1, 2, \dots, (p-1)/2\}$ and $H_p = \{(p+1)/2, \dots, p-1\}$ (the "low" and "high" integers, respectively, between 0 and p). The abbreviations "q.r." and "q.n.r." will stand for "quadratic residue" and "quadratic nonresidue," and Q_p, Q'_p will denote, respectively, the subsets of $\{1, 2, \dots, p-1\}$ consisting of the q.r.'s and q.n.r.'s (mod p).

LEMMA 1. *If $p \in P$, then, for all n, $n \in Q_p \iff p - n \in Q'_p$.*

Proof. In the field $\mathbb{Z}_p$ of integers modulo p, it is well known (see, e.g., [1, Theorem 2.41]) that the group $\mathbb{Z}_p^*$ of nonzero elements is cyclic of order $p-1$, say with generator $[g]$, brackets denoting residue class mod p. (In number-theorists' language, g is a "primitive root" mod p.) Moreover, the only power of $[g]$ having order 2 is $[g]^{(p-1)/2}$, so we must have $[-1] = [g]^{(p-1)/2}$, an *odd* power of $[g]$. It follows that in $\mathbb{Z}_p^*$,

$$\begin{aligned} [n] \text{ is a square} &\iff [n] = [g]^{\text{even}} \iff [-n] = [-1][n] = [g]^{\text{odd}} \\ &\iff [p-n] = [-n] \quad \text{is a nonsquare,} \end{aligned}$$

and the Lemma follows.

LEMMA 2. *If $p \in P$, then $s(Q_p) \equiv s(Q'_p) \equiv 0 \pmod{p}$.*

Proof. As above, the squares in $\mathbb{Z}_p^*$ are $[g^2], [g^4], \dots, [g^{p-1}] = [1]$. Thus, Q_p has $n = (p-1)/2$ elements, and these elements are precisely the roots of the polynomial congruence $x^n - 1 \equiv 0 \pmod{p}$. Hence their sum equals minus the coefficient of x^{n-1} in $x^n - 1$, namely 0 (remember that $p > 3$ so that $n > 1$); that is, $s(Q_p) \equiv 0 \pmod{p}$. But then also $s(Q'_p) \equiv 0 \pmod{p}$, since

$$s(Q_p) + s(Q'_p) = 1 + 2 + \cdots + (p-1) = p(p-1)/2 \equiv 0 \bmod p.$$

Proof of Theorem 2. We have $S_p = s(Q_p \cap L_p) = \Sigma_i(r_i)$ and $S'_p = s(Q'_p \cap L_p) = \Sigma_j(s_j)$ where the r's and s's are the "low" q.r.'s and q.n.r.'s respectively. By Lemma 1, the remaining q.r.'s (the elements of $Q_p \cap H_p$) are the numbers $p - s_j$ and the elements of $Q'_p \cap H_p$ are the numbers $p - r_i$. Thus, letting

$$t = S_p - S'_p = \Sigma_i r_i - \Sigma_j s_j,$$

we have

$$\begin{aligned} s(Q_p \cap L_p) - s(Q'_p \cap L_p) &= t \\ s(Q_p \cap H_p) - s(Q'_p \cap H_p) &= \sum_j (p - s_j) - \sum_i (p - r_i) \\ &= p\big(\#(Q'_p \cap L_p) - \#(Q_p \cap L_p)\big) + t. \end{aligned}$$

Adding these two equations gives

$$s(Q_p) - s(Q'_p) = p\big(\#(Q'_p \cap L_p) - \#(Q_p \cap L_p)\big) + 2t. \tag{2}$$

By Lemma 2, the left side of (2) is congruent to zero (mod p); therefore $t \equiv 0$ (mod p), so $S_p \equiv S'_p$ (mod p), and we can write $t = S_p - S'_p = kp$, $k \in \mathbb{Z}$. This proves the first assertion of Theorem 2. As for the second (concerning the parity of k), we have

$$s(L_p) = 1 + 2 + \cdots + (p-1)/2 = \frac{p-1}{2} \cdot \frac{p+1}{4}$$

and hence

$$\begin{aligned} \frac{p-1}{2} \cdot \frac{p+1}{4} + t &= s(L_p) + t \\ &= \big(s(Q_p \cap L_p) + s(Q'_p \cap L_p)\big) + \big(s(Q_p \cap L_p) - s(Q'_p \cap L_p)\big) \\ &= 2s(Q_p \cap L_p) = 2S_p. \end{aligned} \tag{3}$$

Equation (3) reduces (mod 2) to $(p+1)/4 + t \equiv 0 \bmod 2$, so that t, and hence also $k = t/p$, has the same parity as $(p+1)/4$.

Proof of Theorem 1. Let $p \in P$. The number $r = R(p)$ is the least positive residue (mod p) of $S_p = s(Q_p \cap L_p)$, so $S_p \equiv r$ (mod p). Thus, if we multiply equation (3) by 8 and then reduce mod p (and remember that p divides t), we obtain

$$-1 \equiv 16S_p \equiv 16r \pmod{p}$$

so that p is a divisor of $16r + 1$, say $16r + 1 = pm$. Since $r < p$, we have $m < 16$, and since $p \equiv 3$ (mod 4) we must also have $m \equiv 3$ (mod 4), so that $m \in \{3, 7, 11, 15\}$. Also, if any prime $q > p$ is a divisor of $16r + 1$, then q divides m, so that (since $p \geq 7$) the only possibility is $(p, q) = (7, 11)$. But for $p = 7$ we had $r = 3$ (see table at beginning), and then $16r + 1 = 49$ is *not* divisible by $q = 11$. Hence no such $q > p$ exists, i.e., p is the *largest* prime factor of $16r + 1$ (which in turn implies the statement, "m is the smallest ... "), proving Theorem 1(a). As for part (b), let $r^* \in \mathbb{Z}^+$ and suppose $(16r^* + 1)/m$ is a prime p^* for some $m \in \{3, 7, 11, 15\}$, and assume that m is the least such number that makes this true. Then $m \equiv 3$ (mod 4) implies $p^* \equiv 3$ (mod 4); and if $p^* = 3$, then $p^* m \equiv 1$ (mod 16) implies $m = 11$ (a prime $> p^*$), contradicting "m is the least ... ." Hence $p^* \in P$, so Theorem 1(a) gives $p^* = (16r + 1)/h$ where $r = R(p^*)$ and $h \in \{3, 7, 11, 15\}$. Since $p^* m \equiv 1 \equiv p^* h$ (mod 16), it follows that $m = h$, hence $r^* = r = R(p^*)$.

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Bounding Power Series Remainders

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Most calculus books use the "Lagrange form of the remainder" to bound the truncation error for Taylor series. However, exclusive dependence on this formula has several disadvantages.

1. It works only when one is constructing the Taylor series of a known function, using values of its derivative.
2. It may be very hard or impossible to compute and estimate the derivatives required.
3. The Lagrange remainder is often a poor estimate and is sometimes unusable.

We'll have more to say about these problems, but it is only fair to ask what we propose to use in place of the Lagrange remainder. The answer is surprisingly simple: estimate the tail end of the series by comparison with a *geometric series*. We'll give some examples shortly.

One problem with Lagrange bounds is that they are useful in only one of the cases where power series arise: expanding a known function with known derivatives. This is nice for convincing students that, in principle, their calculators don't actually need massive tables inside them, but can actually *calculate* logs and trig functions. Of course, calculators don't actually *use* Taylor series, but that's another story: in principle they might.

Power series, however, also arise as solutions to differential equations. Until recently this use had been downplayed by many numerical analysts, but new computer algorithms devised by Harley Flanders **[3]** may change that. Arguably, this use of power series is more valuable than the expansion of known functions via Taylor series; however, bounding of the truncation error of such a power series via a Lagrange-type remainder is generally not feasible since the derivatives of these power series are themselves power series.

Geometric series, on the other hand, are relatively simple and straightforward mathematical objects; many students have seen them in high school. Geometric series are perhaps the only series whose convergence properties students really understand; everything is known: exactly when they converge, and what they converge to. They are a solid piece of real-estate in a sea of uncertainty and confusion. Students also seem relatively comfortable with the ratio test, probably because it is easy to remember and use. This should present a good opportunity for discussion of convergence in general by comparison with the geometric case. However, it is unfortunate that those books which do make this comparison rarely continue it into the area of truncation analysis.

It's time for some examples. First let's consider the matter of approximating $e^{.5}$. The error after truncating at the Nth term is, according to Lagrange,

$$\frac{M}{(N+1)!}(.5)^{N+1},$$

where M is a bound on e^x for $0 \le x \le .5$. An easy estimate is $M = 2$, which, for $N = 6$ yields an error of less than 0.0000031.

Now let's look at what's been truncated:

$$
\begin{aligned}
\text{Remainder} &= \frac{.5^7}{7!} + \frac{.5^8}{8!} + \frac{.5^9}{9!} + \frac{.5^{10}}{10!} + \cdots \\
&< \frac{.5^7}{7!} + \frac{.5^8}{7 \cdot 7!} + \frac{.5^9}{7^2 \cdot 7!} + \frac{.5^{10}}{7^3 \cdot 7!} + \cdots \\
&= \frac{.5^7}{7!}\left(1 + \frac{.5}{7} + \frac{.5^2}{7^2} + \frac{.5^3}{7^3} + \cdots\right) \\
&= \frac{.5^7}{7!}\left(\text{sum of a geometric series; first term 1, ratio } \frac{.5}{7}\right) \\
&< \frac{.5^7}{7!}(1.1) < 0.00000171.
\end{aligned}
$$

Not only do we get a better (smaller) bound, but we don't have to worry about how to approximate powers of e as we did with the Lagrange remainder.

Admittedly, both methods here are pretty simple; if anything, the algebra for the Lagrange form is easier. But you don't have to go far to see the algebraic limitations of Lagrange: just try finding and bounding the 7th derivative of arctan x or $\sin(x)/e^x$.

A more spectacular illustration of the shortcomings of the Lagrange remainder occurs in expanding $-\ln(1-x)$ as a Taylor series around 0. It is easily seen that the n^{th} derivative of this function is $-(n-1)!/(1-x)^n$, so

$$-\ln(1-x) = \sum_{K=1}^{N} \frac{x^K}{K} + R_N(x),$$

where

$$R_N(x) = \frac{N!}{(1-\xi)^{N+1}} \cdot \frac{x^{N+1}}{(N+1)!} = \left(\frac{x}{1-\xi}\right)^{N+1} \cdot \frac{1}{N},$$

with $0 \le \xi \le x$.

When $x = 0.75$, the series converges to $\ln 4$, with remainder $\left(\frac{0.75}{1-\xi}\right)^{N+1} \cdot \frac{1}{N}$, where $0 \le \xi \le 0.75$. If we want to know how many terms to take to get a desired accuracy, we have to estimate this remainder—but how? If $\xi > 0.25$ the remainder goes to infinity; otherwise it goes to 0. The Lagrange remainder gives us no useful information.

On the other hand, *without* using the Lagrange remainder, we get, quite simply:

$$
\begin{aligned}
R_N(x) &= \frac{x^{N+1}}{N+1} + \frac{x^{N+2}}{N+2} \cdots \\
&\le \frac{x^{N+1}}{N+1}(1 + x + x^2 + \cdots) \\
&= \frac{x^{N+1}}{N+1}\left(\frac{1}{1-x}\right).
\end{aligned}
$$

When $x = 0.75$ this gives us a usable estimate of the error caused by truncating after N terms.

Now let's look at the differential equations side of the story. The *hyperbolic Bessel function* $I_0(r)$ is a solution to the equation

$$ry''(r) + y'(r) - ry = 0.$$

Fairly easy algebra shows that

$$I_0(r) = \sum_{k=0}^{\infty} \frac{r^{2k}}{4^k (k!)^2}.$$

This series converges for all r, but what is the truncation error if we chop it off after degree $2N$? The remainder is $\sum_{k=N+1}^{\infty} \frac{r^{2k}}{4^k (k!)^2}$, and

$$\frac{\text{term } N+2}{\text{term } N+1} = \frac{r^{2(N+2)}}{4^{N+2}((N+2)!)^2} \cdot \frac{4^{N+1}((N+1)!)^2}{r^{2(N+1)}} = \frac{1}{4}\left(\frac{r}{N+2}\right)^2.$$

Since these ratios get *smaller* with N, the remainder is less than the sum of the geometric series whose first term is "term $N+1$" and which has this last expression as its common ratio. Thus:

$$\text{truncation error} \le \frac{r^{2(N+1)}}{4^{N+1}((N+1)!)^2} \cdot \frac{1}{1 - \frac{1}{4}\left(\frac{r}{N+2}\right)^2}.$$

If there is any other easy way to find such an estimate for the error here, we are not familiar with it; once again the Lagrange remainder won't help.

This kind of error estimation can be introduced very early in the study of series. In fact, when we first compare series in studying convergence, we can point out that whenever a series is dominated by a geometric one, then its remainder is dominated by the corresponding remainder of the geometric series. But *every* convergent power series is, in fact, dominated by a geometric one. (This fact guarantees that we have a radius of convergence.) To see why, suppose that the power series $\sum_{k=0}^{\infty} a_k x^k$ converges for some $x = s$ with $|s| > 0$. Since the terms must be bounded, we have, when $i > N$ say, $|a_i s^i| \le K$ for some K. It follows that $|a_i| \le K/|s|^i$, so we have

$$\left| \sum_{k=N+1}^{\infty} a_k x^k \right| \le \sum_{k=N+1}^{\infty} |a_k x^k| \le K \sum_{k=N+1}^{\infty} \left|\frac{x}{s}\right|^k,$$

which converges for $|x| < |s|$. In our examples, we chose a dominating geometric series that provides a useful bound.

A final question may be posed: How do you prove that various Taylor series converge to the functions they represent without using the remainder term? The solution is simple: invoke the uniqueness theorem for the solutions of differential equations. All elementary functions encountered in calculus satisfy simple differential equations. In fact, it is instructive for students to find such equations for the exponential and trigonometric functions. The Taylor series for these functions can also be shown to satisfy the equations (doing so is a good exercise in manipulation of series).

The intuitive content of the uniqueness theorem for solutions to ordinary differential equations—at least those of orders 1 and 2—is so strong and so important, that it would be a pity if we didn't present it to our students. If there was ever a topic central to "calculus reform," it is this one. Even without a formal analytical proof, its physical and geometric interpretations make acceptance easy and compelling. Furthermore, it is philosophically tied to our perceptions of physical or Newtonian determinism; we invoke it implicitly nearly every time we solve a mechanics problem with calculus.

In summary, we propose de-emphasizing the Lagrange form of the remainder in the curriculum of the usual two-year calculus sequence, and emphasizing instead the use of geometric series bounds on truncation errors, and the introduction of the uniqueness theorem for solutions of differential equations.

We conclude with two notes:

- The ideas in this paper have been tested in the Project CALC program, at Duke University for several years now, and have proved quite successful. A discussion can be found in the Project CALC text **[4]**.
- Comparison with geometric series can be used to *correct* truncation errors by adding on compensating sums. The most famous version of this "acceleration of convergence" technique is called *Aitken's* Δ^2 **[1]**, **[2]**.

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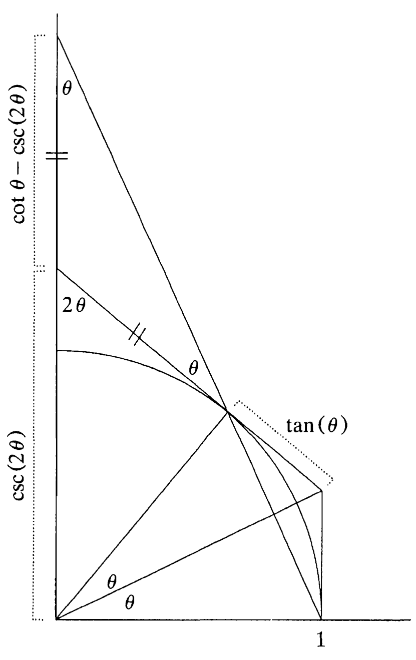
Proof Without Words: Eisenstein's Duplication Formula

$$2\csc(2\theta) = \tan\theta + \cot\theta$$

(G. Eisenstein, *Mathematische Werke*, Chelsea, New York, NY, 1975, page 411.)



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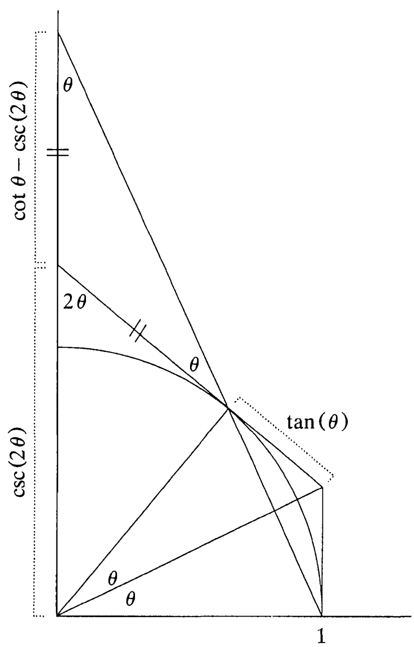

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The Fundamental Theorem of Calculus for Gauge Integrals

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This note presents a Fundamental Theorem of Calculus for gauge integrals that is very general but still accessible to undergraduate students. We first define the gauge integral and give some of its properties. We then introduce an extended notion of differentiation, that of the parametric derivative. Finally, we combine these concepts to state and prove the most general Fundamental Theorem of which we are aware.

The gauge integral The *gauge integral* (also known as the generalized Riemann, or Kurzweil-Henstock integral) is defined using Riemann sums, and is a direct generalization of the Riemann integral (**[4]**, **[5]**, **[6]**). The gauge integral turns out to be more general than the Lebesgue integral as well. The generalization involves a device called a *gauge* which allows the intuitively appealing choice of smaller intervals where the function being integrated is steep, and larger intervals where the function is flat.

As we do for the ordinary Riemann integral, we define a partition of the interval $[a, b]$ to be a set of points $x_0, x_1, x_2, \dots, x_n$ with $a = x_0 < x_1 < \cdots < x_n = b$. We choose a number z_k, called a *tag*, in each interval $[x_{k-1}, x_k]$; the result is a *tagged partition* of the interval $[a, b]$. Then $\sum_{k=1}^{n} f(z_k)(x_k - x_{k-1})$ is a Riemann sum for f on the interval $[a, b]$.

We define a *gauge* γ by choosing for each point p in $[a, b]$ an interval $\gamma(p)$ containing p. A tagged partition is called *γ-fine* if for every k, $1 \le k \le n$, $[x_{k-1}, x_k]$ is a subset of the interval $\gamma(z_k)$. An alternative description of gauge can be given by defining any positive function δ, with domain $[a, b]$, to be a gauge; the equivalence of this to our definition is obtained by letting $\gamma(x) = (x - \delta(x), x + \delta(x))$, for x in $[a, b]$. We leave as an exercise a proof that given any gauge γ on $[a, b]$, there exists a tagged partition of $[a, b]$ that is γ-fine.

The gauge integral is defined as follows. Let f be a real-valued function defined on the interval $[a, b]$. The number I is the *gauge integral* of f on $[a, b]$ if for each positive ϵ there is a gauge γ such that if $\{z_k, [x_{k-1}, x_k]\}$ is a γ-fine tagged partition of $[a, b]$, then

$$\left| I - \sum_{k=1}^{n} f(z_k)(x_k - x_{k-1}) \right| < \epsilon.$$

If γ is a gauge determined by a function whose intervals have lengths bounded away from 0, then the set of tagged partitions is the same as that used for a Riemann integral. Therefore, the gauge integral includes the Riemann integral. The generality of the gauge integral can be seen by some examples.

Example 1. Let $f(x)$ be the Dirichlet function on $[0, 1]$, i.e., $f(x) = 1$ for x rational and $f(x) = 0$ for x irrational. We show that this function has a gauge integral. Let $\epsilon > 0$, and let $\{r_k\}$ be an enumeration of the rationals in $[0, 1]$. Choose a gauge γ

as follows: For each k, let $\gamma(r_k) = \left(r_k - \frac{\epsilon}{2^{k+1}}, r_k + \frac{\epsilon}{2^{k+1}}\right)$. For irrational x, let $\gamma(x) = (-1, 2)$, say. Let $\{z_k, [x_{k-1}, x_k]\}$ be a γ-fine tagged partition of $[0, 1]$. Then if $z_k = r_k$, $f(z_k)(x_k - x_{k-1}) < \frac{\epsilon}{2^k}$. If z_k is not rational, $f(z_x)(x_k - x_{k-1}) = 0$. Therefore

$$\sum_{k=1}^{n} f(z_k)(x_k - x_{k-1}) < \sum_{k=1}^{n} \frac{\epsilon}{2^k} < \epsilon.$$

This shows that the gauge integral of the Dirichlet function is 0. Because f is everywhere discontinuous, it does not have a Riemann integral. Since f is constant except for a set of measure zero (the rationals), however, it does have a Lebesgue integral. Note that the way we selected this gauge corresponds to how one shows that a countable set has Lebesgue measure zero; with a slight modification, this same technique will show that the gauge integral exists for any continuous function whose definition is altered in any bounded way on any countable set.

Example 2. Let $F(x) = x^2 \sin(1/x^2)$ for $x \neq 0$, $F(0) = 0$. This function has a derivative at each point of $[0, 1]$ (the difference quotient is needed to find $F'(0)$). However, the derivative $F'(x)$ is not Riemann integrable on $[0, 1]$, since it is unbounded. The derivative is not Lebesgue integrable either, since $\int_0^1 |F'(x)|\, dx = \infty$. It turns out, however, that $F'(x)$ is gauge integrable, and the Fundamental Theorem of Calculus applies, so

$$\int_0^1 F'(x)\, dx = F(1) - F(0) = \sin 1.$$

This result follows from the theorem proven below.

The Fundamental Theorem of Calculus consists of the equation $\int_a^b f(x)\, dx = F(b) - F(a)$, where F is some type of antiderivative of f on $[a, b]$, together with various hypotheses on f or F. The version of this theorem usually seen in elementary calculus requires that $F'(x) = f(x)$ at each x in $[a, b]$, and that f be continuous on $[a, b]$.

The continuity hypothesis can be relaxed; we need only assume that f is Riemann integrable on $[a, b]$. One proof of this version of the theorem follows from the following chain of equalities, where $x_0, x_1, \dots, x_n$ is a partition of $[a, b]$:

$$F(b) - F(a) = \sum_{k=1}^{n} (F(x_k) - F(x_{k-1})) = \sum_{k=1}^{n} f(z_k)(x_k - x_{k-1}).$$

The first equality is valid because the middle term is a telescoping sum, and the second follows by applying the mean value theorem to F on each subinterval $[x_{k-1}, x_k]$. The last term is a Riemann sum approximating the integral of f. The hypothesis that f is Riemann integrable is satisfied for every bounded function f that is continuous almost everywhere. However, the hypothesis that f has an antiderivative F on $[a, b]$ is difficult to check. For example, a function f with a simple jump discontinuity does not have an antiderivative. (Recall that derivatives must satisfy the intermediate value property, and so cannot have jump discontinuities. See, e.g., [**2**, p. 122].)

For the Lebesgue integral, the Fundamental Theorem of Calculus holds if and only if (i) F is absolutely continuous; and (ii) $F' = f$ almost everywhere on $[a, b]$. One of the most important properties of the gauge integral is that it satisfies an unrestricted form of the Fundamental Theorem of Calculus: if $F' = f$ on $[a, b]$, this theorem holds.

The parametric derivative We turn now to our second main concept, the parametric derivative [**7**]. (We will comment later on relationships between the parametric derivative and the gauge integral.) The idea involved in defining this generalized derivative is not difficult to grasp. In some cases, by parameterizing the independent variable in a non-differentiable function, the composite function is differentiable.

We say that the function F, defined on $[a, b]$, has a *parametric derivative* f on that interval if there exists a strictly increasing differentiable function ϕ, where ϕ maps some interval $[\alpha, \beta]$ onto $[a, b]$, so that $F \circ \phi$ has an ordinary derivative on $[\alpha, \beta]$, with

$$(F \circ \phi)'(t) = f(\phi(t))\phi'(t).$$

Some facts follow from the definition:

1. Since $\phi(t) = t$ is possible, the parametric derivative generalizes the ordinary derivative.
2. If $\phi'(t) \neq 0$, then $F'(x) = f(x)$ is the ordinary derivative at the point $x = \phi(t)$.
3. If the parametric derivative of F is zero at each point of $[a, b]$, then F is constant.
4. Parametric differentiation is a linear operator; thus $(kF)' = kF'$ if k is constant, and $(F + G)' = F' + G'$ (here, primes denote parametric differentiation).

Items 1, 2, and 3 may be checked by the reader. Item 4 is more difficult to prove; for more details, see **[1]**. Observe also that f need not be unique; if $\phi'(t) = 0$, then f can take *any* value at the point $x = \phi(t)$.

Some examples will illustrate the concept of parametric derivative.

Example 3. Let $F(x) = |x|$ on $[-1, 1]$. This function has no derivative at $x = 0$. If $\phi(t) = t^3$, $-1 \leq t \leq 1$, the resulting function $F(\phi(t)) = |t^3|$ is everywhere differentiable. A parametric derivative of F is then f, where $f(x) = -1$ for $x < 0$, $f(x) = 1$ for $x > 0$, and $f(0)$ arbitrary.

Example 4. Let $F(x) = x \sin(1/x)$ for $x \neq 0$, and $F(0) = 0$. This function also fails to have a derivative at $x = 0$, but composing F with $\phi(t) = t^3$ shows that F has a parametric derivative.

These two examples are discussed by A. M. Bruckner as part of general considerations related to "creating" and "destroying" differentiability **[3]**. He describes how a homeomorphic change of variables (composition with ϕ as above) can transform nondifferentiable functions into functions with various differentiability properties. He obtains the following somewhat surprising result:

Example 5. The *Cantor function* is a continuous, nondecreasing function on $[0, 1]$, which is constant on each interval complementary to the Cantor set, and maps the Cantor set onto $[0, 1]$. This function can also be made differentiable by a (highly nontrivial) change of variables. (For details on the Cantor function, see, e.g., [**2**, pp. 135–139] or [**6**, p. 129].)

The main theorem We now proceed to our main theorem. Observe that since ordinary derivatives are also parametric derivatives, the proof is also valid for ordinary derivatives. From now on, integrals will be gauge integrals.

THEOREM. *Let $f(x)$ be a parametric derivative of $F(x)$ on $[a, b]$. Then $f(x)$ is gauge integrable on $[a, b]$, and*

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

We do the main computation of the proof in a lemma.

LEMMA. *Let $F(x)$ have a parametric derivative $f(x)$, with differentiable parametric representation $\phi(t)$ on $[c,d]$. Then, given $p \in [c,d]$ and $\epsilon > 0$, there exists $\delta > 0$ such that if $p - \delta < s < t < p + \delta$, then*

$$|F(\phi(t)) - F(\phi(s)) - f(\phi(p))(\phi(t) - \phi(s))| < \epsilon(t-s).$$

Proof. Using the definition of parametric derivative and standard limit theorems, we get

$$\lim_{t \to p} \left\{ \frac{F(\phi(t)) - F(\phi(p))}{t-p} - f(\phi(p)) \frac{\phi(t) - \phi(p)}{t-p} \right\} = 0.$$

So given $\epsilon > 0$, there exists $\delta = \delta(\epsilon, p) > 0$ such that if $0 < t - p < \delta$, we have

$$\left| \frac{F(\phi(t)) - F(\phi(p))}{t-p} - f(\phi(p)) \frac{\phi(t) - \phi(p)}{t-p} \right| < \epsilon,$$

or, equivalently

$$|F(\phi(t)) - F(\phi(p)) - f(\phi(p))(\phi(t) - \phi(p))| < \epsilon(t-p).$$

Since $0 < p - s < \delta$, it follows from the triangle inequality that

$$\begin{aligned} &|F(\phi(t)) - F(\phi(s)) - f(\phi(p))(\phi(t) - \phi(s))| \\ &\quad = |F(\phi(t)) - F(\phi(p)) + F(\phi(p)) - F(\phi(s)) \\ &\qquad\qquad - f(\phi(p))(\phi(t) - \phi(p)) - f(\phi(p))(\phi(p) - \phi(s))| \\ &\quad \le \epsilon(t-p) + \epsilon(p-s) = \epsilon(t-s). \end{aligned}$$

This completes the proof of the lemma.

To prove the theorem, given a $\phi(t)$ and any $\epsilon > 0$, let $x_0 \in [a,b]$. We note that since $\phi(t)$ is a strictly increasing continuous function, it is a homeomorphism; thus ϕ^{-1} exists and is continuous. If $p_0 = \phi^{-1}(x_0) \in [c,d]$, let $\delta = \delta(\epsilon/(d-c), p_0)$ be given by the lemma. Then, by the continuity of ϕ^{-1} at x_0, there exists a $\delta_1 > 0$, such that if $|x_1 - x_0| < \delta_1$ and $|y_1 - x_0| < \delta_1$, and $t_1 = \phi^{-1}(x_1)$, and $s_1 = \phi^{-1}(y_1)$, then $|t_1 - p_0| < \delta$ and $|s_1 - p_0| < \delta$. Furthermore, if we put $y_1 < x_0 < x_1$, then $s_1 < p_0 < t_1$.

Applying the lemma to s_1, p_0, and t_1 gives

$$|F(\phi(t_1)) - F(\phi(s_1)) - f(\phi(p_0))(\phi(t_1) - \phi(s_1))| < \frac{\epsilon}{d-c}(t_1 - s_1).$$

For each x in $[a,b]$, we choose a $\delta_1(x)$ in this manner. This provides a gauge on $[a,b]$; for each x we choose the open interval $(x - \delta_1(x), x + \delta_1(x))$. Let $D = \{z_i, [x_{i-1}, x_i]: 1 \le i \le n\}$ be a δ_1-fine tagged partition of $[a,b]$. Then

$$\begin{aligned} &\left| \sum_{i=1}^{n} f(z_i)(x_i - x_{i-1}) - (F(b) - F(a)) \right| \\ &\quad = \left| \sum_{i=1}^{n} f(z_i)(x_i - x_{i-1}) - \sum_{i=1}^{n} (F(x_i) - F(x_{i-1})) \right| \\ &\quad = \left| \sum_{i=1}^{n} \{F(x_i) - F(x_{i-1}) - f(z_i)(x_i - x_{i-1})\} \right|. \end{aligned}$$

Define c_i and t_i, $1 \le i \le n$, by $\phi^{-1}(z_i) = c_i$ and $\phi^{-1}(x_i) = t_i$, $1 \le i \le n$. Since x_{i-1} and x_i are contained in $(z_i - \delta_1(z_i), z_i + \delta_1(z_i))$, it follows that t_{i-1} and t_i are contained in $(c_i - \delta(c_i), c_i + \delta(c_i))$. So the last expression becomes

$$\left| \sum_{i=1}^{n} \{F(\phi(t_i)) - F(\phi(t_{i-1})) - f(\phi(c_i))(\phi(t_i)) - \phi(t_{i-1})\} \right|$$
$$\le \sum_{i=1}^{n} |F(\phi(t_i)) - F(\phi(t_{i-1})) - f(\phi(c_i))(\phi(t_i) - \phi(t_{i-1}))|$$
$$\le \sum_{i=1}^{n} \frac{\epsilon}{d-c}(t_i - t_{i-1}) = \epsilon,$$

where the last inequality uses the lemma. This completes the proof of the theorem.

We have found the preceding proof appropriate for advanced calculus. Although the proof is a bit long, the ideas are not difficult, and they offer students a glimpse of some deep ideas in analysis.

Parametric derivatives and gauge integrals Tolstov **[7]** proved that a function $F(x)$ on $[a, b]$ has a parametric derivative $f(x)$ there if and only if $f(x)$ is integrable in the restricted Denjoy sense on $[a, b]$, with $\int_a^b f = F(b) - F(a)$. (Tolstov's result is discussed in **[3**, Theorem 4].) This provides our rationale for the connection between the parametric derivative and the gauge integral in the above theorem, because the gauge integral is equivalent to the restricted Denjoy integral **[5]**. All this implies that the converse of our theorem is true, although we have no elementary proof of this fact. (The main problem is finding a ϕ in the definition of parametric derivative.) It would be interesting to obtain an elementary proof.

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Comparing Sets

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Introduction A finite set A has a size, $|A|$, the number of elements in A. Given any two finite sets A and B, it is intuitively clear that exactly one of the following relations holds: $|A| < |B|$, $|A| = |B|$, $|B| < |A|$. This is so because the set $\mathbb{Z}^+$ of non-negative integers satisfies the *trichotomy property*: for $m, n \in \mathbb{Z}^+$ we have $n < m$, $n = m$, *or* $m < n$, and each of these relations excludes the other two. Cantor extended the notion of the size of a set by introducing *cardinal numbers*. Two sets A and B have the same cardinality (i.e., size), and we write $|A| = |B|$, if there exists a one-to-one and onto function f: $A \to B$. If there exists a one-to-one function f: $A \to B$, but $|A| \neq |B|$, then $|A| < |B|$. It is certainly desirable that the cardinalities of two sets A, B be uniquely comparable, i.e., that precisely one of the relations $|A| < |B|$, $|A| = |B|$, or $|B| < |A|$ holds. In the beginning of his famous *Beiträge* **[1, 2, 3]**, Cantor gave a simple argument showing that at most one of the relations $\alpha < \beta$, $\alpha = \beta$, and $\beta < \alpha$ holds for any two cardinal numbers α and β **[3**, pp.89–90**]**. He then wrote:

> On the other hand the theorem that, with any two cardinal numbers α and β, one of those three relations must be realized is by no means self evident, and can hardly be proved at this stage.

After this statement, Cantor claimed that the theorem would be proved at a later stage. It wasn't proved at a later stage.

Although well ordered sets and ordinal numbers were defined and studied by Cantor in the second part of the *Beiträge* **[2]**, he proved trichotomy for ordinals in section 13 of the first part **[1, 3]**. (Well ordered sets and ordinal numbers will be defined later in this note.) In **[7, 8]**, Zermelo proved that every set can be well ordered, and pointed out that this fact, coupled with ordinal trichotomy, implies trichotomy for cardinal numbers.

A simple, straightforward proof of cardinal trichotomy, using Zorn's lemma, (see **[5**, section 18**]** for a statement and proof of Zorn's lemma) is as follows: Let A and B be two sets with $|B| \not\leq |A|$. A function f: $A' \to B$ with $A' \subseteq A$ is called a *partial function* from A to B. Let $\mathscr{F}$ be the set of all one-to-one partial functions from A to B. Let f_1: $A_1 \to B$ and f_2: $A_2 \to B$ be two partial functions from A to B, with $A_1 \subseteq A_2$. If the restriction of f_2 to A_1 is f_1, then f_2 is called an *extension* of f_1, and we write $f_1 \leq f_2$. It is easy to see that $\mathscr{F}$ is a partially ordered set under the relation $\leq$. By Zorn's lemma, there exists a maximal element $f \in \mathscr{F}$. Let A' be the domain of f, and let B' be the range of f. If $B' = B$, then $|B| \leq |A|$, a contradiction. If $A' = A$ then $|A| \leq |B|$. If $A' \subset A$ and $B' \subset B$, then choose $a \in A \setminus A'$ and $b \in B \setminus B'$. Define f': $A \cup \{a\} \to B$ by

$$f'(x) = \begin{cases} f(x) & \text{if } x \in A' \\ b & \text{if } x = a. \end{cases}$$

Then $f' \in \mathscr{F}$, and $f < f'$, a contradiction.

In this note, the method just used to prove trichotomy for *cardinal* numbers will be employed to prove *ordinal* trichotomy. The set $\mathscr{F}$ will be taken to be a set of certain order-preserving functions, and again Zorn's lemma will yield the result. There is a pedagogical advantage in this proof. A beginner in set theory can first see the method of proof in the easier cardinal case, and then better understand the proof of the more complicated ordinal case. (It is interesting to note that it was easier for Cantor to deal with trichotomy for ordinals than for cardinals. He never succeeded in proving cardinal trichotomy.)

The trichotomy theorem for ordinals was also proved by Baire, Hausdorff, and others. Hausdorff's proof is especially attractive (see the Comparability Theorem in **[6]**). The proof relies on the fact that the set $W(\alpha)$, consisting of all ordinals less than a given ordinal α, is a well ordered set with ordinality α.

Much information about Cantor and his work may be found in **[4]**.

Ordinal Trichotomy A partially ordered set is said to be *well ordered* if every non-empty subset of A possesses a smallest element.

All sets in this paper will be assumed to be well ordered.

The smallest element of a set A will be denoted by $\min(A)$. For $a \in A$, the set $s(a) = \{x \in A | x < a\}$ is called the *initial segment* of a. A function $f\colon A \to B$ is said to be *order preserving* if $f(a_1) < f(a_2)$ for all a_1, a_2 satisfying $a_1 < a_2$. If an order preserving function $f\colon A \to B$ is onto, then f is called a *similarity*, and the sets A and B are said to be *similar*, and we write $A \simeq B$. Two sets A and B have the same ordinality, $\operatorname{ord}(A) = \operatorname{ord}(B)$, if $A \simeq B$. If $A \simeq B'$ for some subset $B' \subset B$, but $A \not\simeq B$, then $\operatorname{ord}(A) < \operatorname{ord}(B)$. The domain of a function f will be denoted $\operatorname{dom}(f)$. A set A is a *section* of a set B if $A \simeq s(b)$ for some $b \in B$.

The following propositions are well known and easy to prove (see, e.g., [**5**, p.67 and p.72]).

PROPOSITION 1. (Transfinite induction) *Let* $A' \subseteq A$. *If* $s(a) \subseteq A' \Rightarrow a \in A'$ *for all* $a \in A$, *then* $A' = A$.

PROPOSITION 2. *Let* $f\colon A \to A$, *be order preserving. Then* $a \leq f(a)$ *for all* $a \in A$.

PROPOSITION 3. *Let* $f\colon A \to B$, *be a similarity. Then* $f\colon s(a) \to s(f(a))$ *is a similarity for all* $a \in A$.

A simple consequence of Proposition 2 is the following result of Cantor [**3**, p.144], which he called B:

THEOREM B. *Let* $a \in A$. *Then* $A \not\simeq s(a)$.

Theorem B implies the following corollary:

LEMMA 4. *For two sets* A *and* B, *at most one of the following conditions is satisfied*: A *is a section of* B; $A \simeq B$; B *is a section of* A.

Let $A' \subseteq A$ be such that either $A' = A$ or $A' = s(a)$ for some $a \in A$. A function $f\colon A' \to B$ satisfying $f(a) = \min(B \setminus f(s(a))$ for all $a \in A'$ is called a *partial similarity* from A to B, or simply a partial similarity.

Let $f\colon A \to B$ be a similarity, let $a \in A$, and let $b \in B \setminus f(s(a))$. Since f is onto, there exists $a' \in A$ such that $f(a') = b$. Since $b \notin f(s(a))$, it follows that $a' \notin s(a)$, i.e., $a \leq a'$. Hence $f(a) \leq f(a')$, and so $f(a) = \min(B \setminus f(s(a))$. This shows that every similarity is a partial similarity.

LEMMA 5. *Let* $f, g\colon A' \to B$ *be partial similarities. Then* $f = g$.

Proof. Let $A'' = \{a \in A' | f(a) = g(a)\}$. In order to show that $A'' = A'$ it suffices to show, by Proposition 1, that for $a \in A'$ the assumption $s(a) \subseteq A''$ implies that $a \in A''$, or equivalently, that $f(a) = g(a)$. Now $f(a) = \min(B \setminus f(s(a))) = \min(B \setminus g(s(a))) = g(a)$.

LEMMA 6. *Let $f: A' \to B$ be a similarity. Then $f: A' \to f(A')$ is a partial similarity.*

Proof. Let $a_1, a_2 \in A'$, with $a_1 < a_2$. Since $f(a_1) \in f(s(a_2))$ and $f(a_2) \notin f(s(a_2))$, it follows that $f(a_1) \neq f(a_2)$. It therefore suffices to show that $f(a_1) \leq f(a_2)$. The inclusion $s(a_1) \subset s(a_2)$ implies that $B \setminus f(s(a_2)) \subseteq B \setminus f(s(a_1))$, which yields that $f(a_1) = \min(B \setminus f(s(a_1))) \leq \min(B \setminus f(s(a_2))) = f(a_2)$.

LEMMA 7. *Let $f: A' \to B$ be a partial similarity. Then either $f(A') = B$ or $f(A') = s(b)$ for some $b \in B$.*

Proof. Suppose that $f(A') \neq B$. Let $b = \min(B \setminus f(A'))$. Clearly, $s(b) \subseteq f(A')$. Let $a' \in A'$. Then $s(a') \subset A' \Rightarrow B \setminus f(A') \subset B \setminus f(s(a'))$. Taking the minimum of these sets shows that $f(a') \leq b$, and from the definition of b, it is clear that $f(a') \neq b$. Therefore, $f(a') < b$, and so $f(A') = s(b)$.

An immediate consequence of Lemmas 6 and 7 is as follows:

COROLLARY 8. *Let f: $A' \to B$ be a partial similarity. Then either $A' \simeq B$, or A' is a section of B.*

Let $\mathscr{F}$ denote the set of partial similarities from A to B, and let $f_1 \leq f_2$ denote the relation that f_2 is an extension of f_1. It is readily seen that $\mathscr{F}$ is a partially ordered set under this relation.

LEMMA 9. *Let $f_1, f_2 \in \mathscr{F}$. If $\operatorname{dom}(f_1) \subseteq \operatorname{dom}(f_2)$ then f_2 is an extension of f_1.*

Proof. If $\operatorname{dom}(f_1) = \operatorname{dom}(f_2)$ then $f_1 = f_2$, by Lemma 5. It may therefore be assumed that $\operatorname{dom}(f_1) = s(a)$, with $a \in \operatorname{dom}(f_2) = A'$. Now f_2: $A' \to f_2(A')$ is a similarity, so f_2: $s(a) \to s(f_2(a))$ is also a similarity, by Proposition 3. Since similarities are partial similarities, it follows that both f_1, f_2:$\operatorname{dom}(f_1) \to B$ are partial similarities. Now Lemma 5 implies that the restriction of f_2 to $\operatorname{dom}(f_1)$ is f_1, i.e., $f_1 \leq f_2$.

Lemma 9 and the fact that A is well ordered yields that $\mathscr{F}$ is well ordered.

LEMMA 10. *$\mathscr{F}$ possesses a maximal element.*

Proof. Consider the function f_0: $\{\min(A)\} \to B$ defined by $f_0(\min(A)) = \min(B)$. Since f_0 belongs to $\mathscr{F}$, $\mathscr{F} \neq \varnothing$. Let $C = \{f_i | i \in I\}$ be a chain in $\mathscr{F}$, and let $A' = \cup_{i \in I} \operatorname{dom}(f_i)$. Define f: $A' \to B$ by $f(a) = f_i(a)$ for $a \in \operatorname{dom}(f_i)$. Lemma 9 assures that f is well defined. Clearly f is an extension of f_i for each $i \in I$. To show that f is an upper bound for C in $\mathscr{F}$ it suffices to show that either $A' = A$ or $A' = s(a)$ for some $a \in A$, and that f is a partial similarity. If $\operatorname{dom}(f_i) = A$ for some $i \in I$, then $f = f_i$ is a partial similarity. It may therefore be assumed that for each $i \in I$, $\operatorname{dom}(f_i) = s(a_i)$ for some $a_i \in A$. Suppose that $A' \neq A$. Let $a = \min(A \setminus A')$. Clearly, $s(a) \subseteq A'$. Let $a' \in A'$. Then $a' \in s(a_i)$ for some $i \in I$. If $a \leq a'$ then $a \in s(a_i) \subseteq A'$, a contradiction. Therefore $a' \in s(a)$, and so $A' = s(a)$. Now

$$f(a') = f_i(a') = \min(B \setminus f_i(s(a'))) = \min B \setminus f(s(a')),$$

and so f is a partial similarity. By Zorn's lemma $\mathscr{F}$ possesses a maximal element.

THEOREM 11. *Precisely one of the following conditions is satisfied: A is a section of B; $A \simeq B$; B is a section of A.*

Proof. By Lemma 4, it suffices to show that at least one of the above conditions is satisfied. Let f: $A' \to B$ be a maximal element in $\mathscr{F}$. There are three cases to consider: (1) $A' = A$; (2) $A' \neq A$ and $f(A') = B$; (3) $A' \neq A$ and $f(A') \neq B$. In case (1), Corollary 8 yields that either $A \simeq B$ or A is a section of B. Clearly, $B \simeq A'$ in case (2).

Since in this case $A' = s(a)$ with $a \in A$, it follows that B is a section of A. In case (3), $A' = s(a)$ for some $a \in A$. Let $b = \min(B \setminus f(A'))$. Define $\hat{f}$: $A' \cup \{a\} \to B$ by

$$\hat{f}(x) = \begin{cases} f(x) & \text{if } x \in A' \\ b & \text{if } x = a. \end{cases}$$

It is readily seen that $\hat{f} \in \mathscr{F}$, and that $f < \hat{f}$; this contradiction completes our proof of ordinal trichotomy.

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Inclusion-Exclusion and Characteristic Functions

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Introduction The *inclusion-exclusion formula* of set theory and combinatorics concerns the number of elements in unions of finite sets. In the simplest case, it takes the form

$$n(A \cup B) = n(A) + n(B) - n(A \cap B), \tag{1}$$

where A and B are subsets of some universal set S, and $n(X)$ denotes the number of elements in a set X. In the case of three subsets A, B, and C, we have

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C).$$

The general formula can be written as follows:

$$n(A_1 \cup A_2 \cup \cdots \cup A_p) = \sum_{1 \le i \le p} n(A_i) - \sum_{1 \le i < j \le p} n(A_i \cap A_j) + \sum_{1 \le i < j < k \le p} n(A_i \cap A_j \cap A_k) - \cdots + (-1)^{p+1} n(A_1 \cap A_2 \cap \cdots \cap A_p). \tag{2}$$

Although sometimes classified as an advanced counting technique (see, e.g., **[2]**), the formula is a fairly elementary result. When the number of subsets involved is small, Since in this case $A' = s(a)$ with $a \in A$, it follows that B is a section of A. In case (3), $A' = s(a)$ for some $a \in A$. Let $b = \min(B \setminus f(A'))$. Define $\hat{f}$: $A' \cup \{a\} \to B$ by

$$\hat{f}(x) = \begin{cases} f(x) & \text{if } x \in A' \\ b & \text{if } x = a. \end{cases}$$

It is readily seen that $\hat{f} \in \mathscr{F}$, and that $f < \hat{f}$; this contradiction completes our proof of ordinal trichotomy.

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where A and B are subsets of some universal set S, and $n(X)$ denotes the number of elements in a set X. In the case of three subsets A, B, and C, we have

$$\begin{aligned} n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) \\ - n(C \cap A) + n(A \cap B \cap C). \end{aligned}$$

The general formula can be written as follows:

$$\begin{aligned} &n(A_1 \cup A_2 \cup \cdots \cup A_p) \\ &\quad = \sum_{1 \le i \le p} n(A_i) - \sum_{1 \le i < j \le p} n(A_i \cap A_j) + \sum_{1 \le i < j < k \le p} n(A_i \cap A_j \cap A_k) - \cdots \\ &\qquad + (-1)^{p+1} n(A_1 \cap A_2 \cap \cdots \cap A_p). \end{aligned} \tag{2}$$

Although sometimes classified as an advanced counting technique (see, e.g., [**2**]), the formula is a fairly elementary result. When the number of subsets involved is small, say 2 or 3, its truth can be seen immediately by means of a Venn diagram. It is nonetheless an important general principle of combinatorics, used, for instance, to find the number of derangements of n objects. (A *derangement* is a permutation in which no element occupies its original place.)

In [2] the formula is proved by showing that "an element in the union is counted exactly once by the right-hand side of the equation." A similar proof is given in [1]. Another approach is to apply induction, after establishing the intuitively obvious case of two sets.

We would like to draw attention to another proof, which we think deserves to be better known (the author could not find it in any of his numerous books on discrete mathematics). The main idea of the proof is certainly not new (we welcome references).

The characteristic function We begin by defining our main tool and stating some of its basic properties. For $A \subset S$, the *characteristic function* q_A: $S \to \{0, 1\}$ is defined by

$$q_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

The cardinality (number of elements) of A can now be written as

$$n(A) = \sum_{x \in S} q_A(x). \tag{3}$$

We will also need the formulas

$$q_{A \cap B} = q_A q_B \tag{4}$$

$$q_{\bar{A}} = \mathbf{1} - q_A \tag{5}$$

where $\bar{A}$ is the complement of A, and $\mathbf{1}$ is the *unit function*: $\mathbf{1}(x) = 1$ for all $x \in S$.

The formulas (4) and (5) are easily checked: $(q_A q_B)(x) = q_A(x) q_B(x)$ is 1 exactly when x belongs to both A and B; and $(\mathbf{1} - q_A)(x) = \mathbf{1}(x) - q_A(x) = 1 - q_A(x)$ is 1 exactly when $q_A(x)$ is 0, i.e., when $x \in \bar{A}$.

Proving the inclusion-exclusion formula Let us first attack the simple case with two subsets. To begin with, a suitable expression for the characteristic function of $A \cup B$ is derived:

$$\begin{aligned} q_{A \cup B} &= \mathbf{1} - q_{\overline{A \cup B}} && \text{(by (5))} \\ &= \mathbf{1} - q_{\bar{A} \cap \bar{B}} && \text{(by de Morgan's law)} \\ &= \mathbf{1} - q_{\bar{A}} q_{\bar{B}} && \text{(by (4))} \\ &= \mathbf{1} - (\mathbf{1} - q_A)(\mathbf{1} - q_B) && \text{(by (5))} \\ &= \mathbf{1} - (\mathbf{1} - q_A - q_B + q_A q_B) \\ &= q_A + q_B - q_A q_B = q_A + q_B - q_{A \cap B}. && \text{(by (4))} \end{aligned}$$

Using equation (3), we now obtain formula (1):

$$\begin{aligned} n(A \cup B) &= \sum_{x \in S} q_{A \cup B}(x) = \sum_{x \in S} (q_A + q_B - q_{A \cap B})(x) \\ &= \sum_{x \in S} \{q_A(x) + q_B(x) - q_{A \cap B}(x)\} \\ &= \sum_{x \in S} q_A(x) + \sum_{x \in S} q_B(x) - \sum_{x \in S} q_{A \cap B}(x) \\ &= n(A) + n(B) - n(A \cap B). \end{aligned}$$

A similar proof can be carried out in the general case:

$$\begin{aligned} q_{A_1 \cup \cdots \cup A_p} &= \mathbf{1} - q_{\overline{A_1 \cup \cdots \cup A_p}} = \mathbf{1} - q_{\overline{A_1} \cap \cdots \cap \overline{A_p}} \\ &= \mathbf{1} - (\mathbf{1} - q_{A_1}) \cdots (\mathbf{1} - q_{A_p}) \\ &= \sum_{1 \le i \le p} q_{A_i} - \sum_{1 \le i < j \le p} q_{A_i} q_{A_j} + \sum_{1 \le i < j < k \le p} q_{A_i} q_{A_j} q_{A_k} - \cdots \\ &\quad + (-1)^{p+1} q_{A_1} q_{A_2} \cdots q_{A_p} \\ &= \sum_{1 \le i \le p} q_{A_i} - \sum_{1 \le i < j \le p} q_{A_i \cap A_j} + \sum_{1 \le i < j < k \le p} q_{A_i \cap A_j \cap A_k} - \cdots \\ &\quad + (-1)^{p+1} q_{A_1 \cap A_2 \cdots \cap A_p}, \end{aligned}$$

from which (2) is obtained by means of (3).

Remarks Our proof develops the formula "from first principles," whereas in approaches like the one in [**2**], the formula has to be known at the outset as a guess or hypothesis, and the proof consists of a verification of the hypothesis. On the other hand, our proof cannot be fully understood and appreciated without a modest amount of familiarity with abstract algebra or functional analysis (the fascinating insight that *functions* can be treated as elements of an algebra—added, multiplied, etc.). This may, unfortunately, limit the proof's usefulness in basic courses.

We note, finally, that the finiteness of the set S is not essential; one can just replace the sum in (3) by a suitable integral. The finiteness of the number of subsets considered is more difficult (impossible?) to get around.

A generalization can be achieved by introducing a weight function $w(x)$, and replacing $n(A)$ by a more general "magnitude"

$$m(A) = \sum_{x \in S} w(x) q_A(x).$$

Acknowledgment. The author is indebted to the referee for valuable remarks and suggestions.

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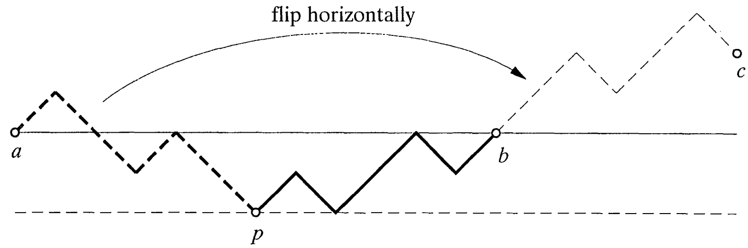
Proof Without Words: Bijection Between Certain Lattice Paths

Dedicated to Ernst Specker on the occasion of his 78th birthday.



ab: lattice path with starting point and endpoint on the same (given) level
p: first minimum on the path ab
pc: lattice path staying above initial level (non-ruin path)

Conclusion There exist as many non-ruin paths of length $2n$ as paths of length $2n$ with starting point and endpoint on the same level, namely $\binom{2n}{n}$.

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An Antisymmetric Formula for Euler's Constant

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The formula

$$\gamma = \lim_{x \to 1^+} \sum_{n=1}^{\infty} \left(\frac{1}{n^x} - \frac{1}{x^n} \right) \tag{1}$$

shows that Euler's constant, γ, which is defined (see [**1**]) by

$$\gamma = \lim_{n \to \infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n \right), \tag{2}$$

is the limit as x approaches 1 from above of a series whose terms are antisymmetric in n and x. The formula also implies that γ is the limit as $x \to 1^+$ of the difference between the p-series $\sum_{n=1}^{\infty} 1/n^x$ and the geometric series $\sum_{n=1}^{\infty} 1/x^n$, because

$$\sum_{n=1}^{\infty} \left(\frac{1}{n^x} - \frac{1}{x^n} \right) = \sum_{n=1}^{\infty} \frac{1}{n^x} - \sum_{n=1}^{\infty} \frac{1}{x^n}$$

for $x > 1$. On the other hand, since the geometric series sums to $1/(x-1)$, the formula is itself an immediate consequence of the fact (see [**4**, Section 2.1]) that

$$\lim_{x \to 1} \left(\zeta(x) - \frac{1}{x-1} \right) = \gamma,$$

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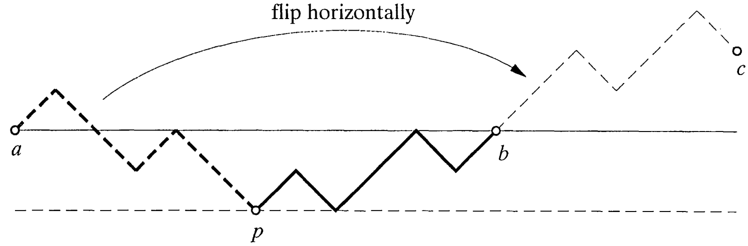

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where $\zeta(x) = \sum_{n=1}^{\infty} 1/n^x$ is the Riemann zeta function. (For a connection between γ

and the zeros of the zeta function, as well as a wealth of other information and references on γ, see [**2**].)

We now give an independent proof of the formula. First, note that

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \log(n+1)\right)$$

is equivalent to the definition of γ, because

$$\lim_{n\to\infty}(\log n - \log(n+1)) = \lim_{n\to\infty}\log\left(\frac{n}{n+1}\right) = 0.$$

Now write

$$\frac{1}{x-1} = \int_1^\infty \frac{dt}{t^x} = \sum_{n=1}^{\infty}\int_n^{n+1}\frac{dt}{t^x}$$

and

$$\log(n+1) = \int_1^{n+1}\frac{dt}{t} = \sum_{k=1}^{n}\int_k^{k+1}\frac{dt}{t}$$

as sums of integrals. It follows that the limits in equations (1) and (2) can be written as

$$\lim_{x\to 1^+}\sum_{n=1}^{\infty}\left(\frac{1}{n^x} - \int_n^{n+1}\frac{dt}{t^x}\right) \tag{3}$$

and

$$\sum_{n=1}^{\infty}\left(\frac{1}{n} - \int_n^{n+1}\frac{dt}{t}\right),$$

respectively. The two limits are therefore the same, since the latter series is the term-by-term limit of the former series, which we now show converges uniformly, so that interchanging the limit and the summation is justified. To prove uniform convergence of the series in formula (3) on the interval $[1,2]$, we apply the Weierstrass M-test (see, e.g., [**3**]), using the series Σn^{-2} for comparison:

$$0 < \frac{1}{n^x} - \int_n^{n+1}\frac{dt}{t^x} = \int_n^{n+1}\left(\frac{1}{n^x} - \frac{1}{t^x}\right)dt = \int_n^{n+1}\left(\int_n^t xu^{-x-1}\,du\right)dt$$

$$\le xn^{-x-1}\int_n^{n+1}\left(\int_n^t du\right)dt = \frac{1}{2}xn^{-x-1} \le n^{-2}$$

for $1 \le x \le 2$. This completes the proof of the antisymmetric formula for Euler's constant.

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Triangles with the Same Centroid

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Introduction A well known fact in Euclidean geometry is that the medians of a triangle intersect in one point, called the centroid. If G is the centroid of the triangle ABC, then G divides each of the medians AL, BN, and CK in the ratio $2:1$. This property is used to derive the algebraic relation, $G = \frac{1}{3}(A + B + C)$, where A, B, C, and G are treated here as numbers in the complex plane [3]. Another known property of G, is that it is also the centroid of the medial triangle KLN [**1**], [**2**] (Figure 1a). In his *Advanced Euclidean Geometry* [**4**, p. 175], Roger Johnson brings to our attention the following pleasant result concerning G.

Theorem 1. *If the vertices of a triangle lie on the sides of another, and divide them in a fixed ratio, the triangles have the same centroid* G (Figure 1b).

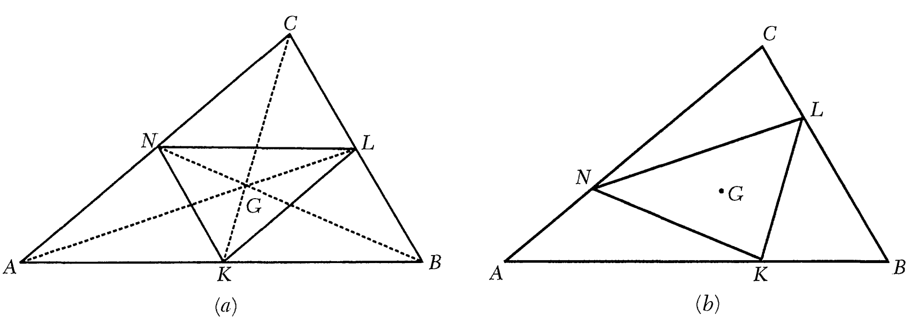

FIGURE 1

Johnson provides a synthetic proof, which he attributes to the German geometer Wilhelm Fuhrmann (1833–1904).

Generalization In this note, we generalize Theorem 1 as follows:

Theorem 2. *If on the sides of an arbitrary triangle ABC three similar triangles AKB, BLC, and CNA are drawn outward (or inward), then the triangles KLN and ABC have the same centroid* G (Figure 2a *and* 2b).

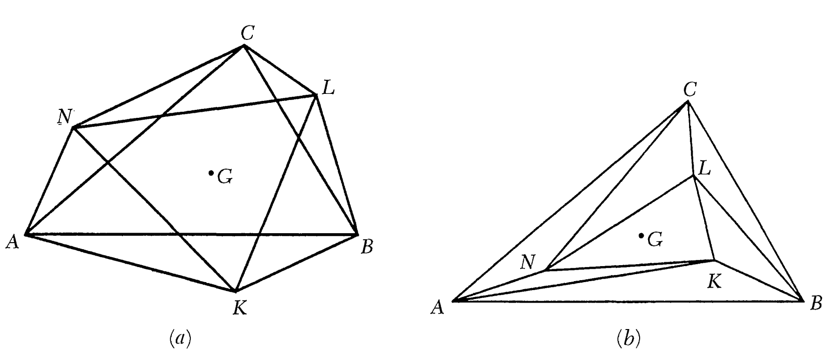

FIGURE 2

This theorem may be interpreted in two different ways. The *synthetic* interpretation is that the six medians, three of each of the triangles KLN and ABC, concur in the point G. The *analytic* interpretation is that the arithmetic mean of the complex numbers representing the vertices K, L, and N is the same as that of the vertices A, B, and C. We will prove Theorem 2 using complex arithmetic.

Consider the points A, B, C, K, L, and N as numbers in the complex plane. The similarity of the triangles AKB, BLC, and CNA implies that for some fixed complex number $Z \neq 0$, $K - A = Z(B - A)$, $L - B = Z(C - B)$, and $N - C = Z(A - C)$. If these equations are added, it follows immediately that $K + L + N = A + B + C$, which completes the proof. The triangles AKB, BLC, and CNA will be outward (or inward) relative to the triangle ABC according to $-\pi < \arg(Z) < 0$ (or $0 < \arg(Z) < \pi$). When $\arg(Z) = 0$ or $\arg(Z) = \pi$, each of the triangles will degenerate into collinear line segments. In particular, if $\arg(Z) = 0$ and $|Z| < 1$, the points K, L, and N will divide AB, BC, and CA respectively in the same ratio and this represents the case of Theorem 1.

Applications First, let us draw the three squares AA_1B_2B, BB_1C_2C, and CC_1A_2A externally on the sides AB, BC, and CA of an arbitrary triangle ABC. Denote the centers of the squares by K, L, and N respectively. Then by Theorem 2, the triangles ABC, KLN, $A_1B_1C_1$, and $A_2B_2C_2$ all have the same centroid G (see FIGURE 3).

Second, suppose we draw equilateral triangles AC_1B, BA_1C, and CB_1A externally on the sides of the triangle ABC. Denote the centroids of the equilateral triangles by K, L, and N respectively (see FIGURE 4).

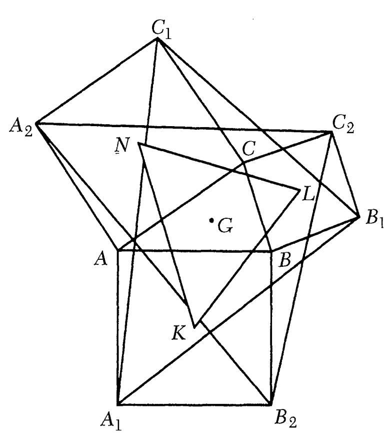

FIGURE 3

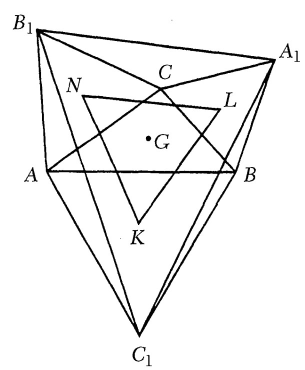

FIGURE 4

Theorem 2 implies that the triangles ABC, $A_1B_1C_1$, and KLN have the same centroid, G. The triangle KLN is sometimes called *Napoleon's triangle*. The French general is said to have proved that it is equilateral **[2]**, **[3]**, and **[5]**.

The converses The converse of Theorem 1 asserts that if one triangle is inscribed in another, so that both have the same centroid, then the vertices of the former divide the sides of the latter in equal ratios. Johnson hints that this converse can be proved by reversing Fuhrmann's proof of Theorem 1.

The converse of Theorem 2 may be stated as follows: If on the sides of an arbitrary triangle ABC, three triangles AKB, BLC, and CNA are drawn outward (or inward), such that the triangles KLN and ABC have the same centroid, then the triangles AKB, BLC, and CNA are similar. We will give a counterexample to show that this converse is false.

Let ABC be a scalene triangle with centroid G and circumcenter O (see FIGURE 5). Take an arbitrary point K on the smaller arc AB of the circumcircle of triangle ABC. Join KG and extend it to E such that $GE = \frac{1}{2}KG$. Join OE and erect on it at E a perpendicular that meets the circumcircle at L (on arc BC) and N (on arc AC).

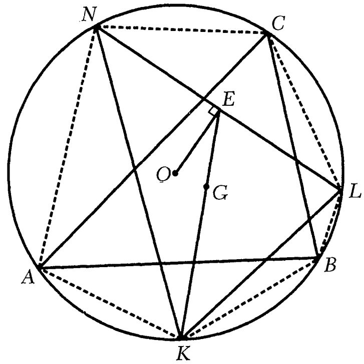

FIGURE 5

It is easy to see that triangle KLN has G as its centroid. Hence triangles ABC and KLN have the same centroid. However, the triangles AKB, BLC, and CNA are not similar, because the angles AKB, BLC, and CNA are inscribed in unequal circular segments.

Triangles with the same nine-point circle The circle that passes through the midpoints of the sides of a triangle is called its nine-point circle [**2**]. In FIGURE 6, triangles ABC and KLN are inscribed in a circle with center O and have the same

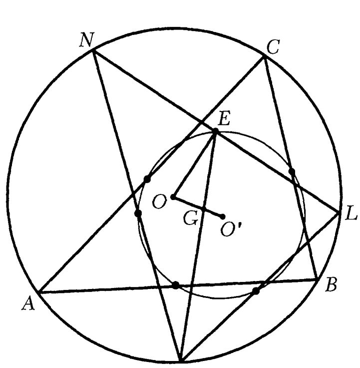

FIGURE 6

centroid G. Consequently, the two triangles have the same nine-point circle whose center O' divides OG externally in the ratio $3:1$ [**1**]. Now, if K moves along the circumcircle of the triangle ABC, then the infinitely many triangles such as KLN will have a common nine-point circle traced by the point E. In 1822, Karl Feuerbach of Germany proved that the nine-point circle of a triangle touches the incircle and the three excircles of the triangle [**1**], [**4**]. This theorem, together with the existence of infinitely many triangles sharing the same nine-point circle, implies that this circle touches infinitely many incircles and excircles.

Acknowledgment The author would like to thank the referees for their valuable suggestions.

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2. H. S. M. Coxeter and S. L. Greitzer, *Geometry Revisited*, New Mathematical Library (Volume 19), Random House Inc., New York, NY, 1967.
3. Liang-shin Hahn, *Complex Numbers and Geometry*, The Mathematical Association of America, Washington, DC, 1994.
4. R. A. Johnson, *Advanced Euclidean Geometry*, Dover Publication, New York, NY, 1960.
5. I. M. Yaglom, *Geometric Transformations I*, New Mathematical Library (Volume 8), Random House Inc., New York, NY 1962.

Editor's Notes

Articles in the December 1997 MAGAZINE drew several letters. Concerning a Proof Without Words (page 380), Harold Boas wrote:

> Your readers may be interested to know that the PWW showing the area of a right triangle to be equal to the sum of the areas of two lunes... was discovered in the 5th century BC by Hippocrates (the mathematician, not the physician).... The fame of Hippocrates indeed rests largely on his quadrature of lunes, the first rigorous determinations of areas of curves regions: see, for example, chapter 1 of the 1990 book *Journey Through Genius*, by William Dunham.

Several readers commented on *The truel* (pp. 315–326), by D. M. Kilgour and S. J. Brams. Harold Boas wrote:

> The article... by D. M. Kilgour and S. J. Brams rekindled a fond childhood memory: my father reading aloud A. P. Herbert's comic drama *Fat King Mellon and Princess Caraway*. Scene III features a humorous encounter in which the King (traveling incognito) aims his blunderbuss at the Princess (in disguise), who draws her bow at the highwayman, who in turn covers the King with his pistol. All fire at once, and the universally fatal results necessitate the intervention, *deux ex machina*, of the Fairy Gurgle to permit action to continue. The earliest (admittedly nonmathematical) reference I know to truels is this play, written for the 1924 birthday of a ten-year-old girl, and published in 1927 by Oxford University Press.

Editor's Notes continue on page 231

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Editor's Notes continue on page 231

Answers

Solutions to the Quickies on page 226.

A880. The right-hand side is clearly the number of one-to-one functions from an n-element set to itself.

Since there are $(n-k)^n$ functions from an n-element set to an $(n-k)$-element set, and an n-element set has $\binom{n}{k}$ subsets with $(n-k)$ elements, the left-hand side is the number of onto functions from an n-element set to itself (by the inclusion-exclusion principle).

Finally, since the one-to-one functions and the onto functions from an n-element set to itself are identical, equality follows.

A881. The only four such functions are

$$f(x,y)=x, f(x,y)=y, f(x,y)=\min\{x,y\}, f(x,y)=\max\{x,y\}.$$

Clearly $f(x,y)=x=y$ on the line $x=y$. Let $A=\{(x,y): f(x,y)=x\}$. Then A is a closed set and its complement A^c is the open set $A^c=\{(x,y): f(x,y)=y, x\neq y\}$. Similarly, $B=\{(x,y): f(x,y)=y\}$ is closed and $B^c=\{(x,y): f(x,y)=x, x\neq y\}$ is open. The open half-plane $U=\{(x,y): x>y\}$ is a connected set. Since U is the disjoint union of the open sets $U\cap A^c$ and $U\cap B^c$, either $U\cap A^c=\varnothing$ or $U\cap B^c=\varnothing$. Thus, either $f(x,y)=x$ or $f(x,y)=y$ on U. Similarly, either $f(x,y)=x$ or $f(x,y)=y$ on $\{(x,y): x<y\}$. The four combinations of possibilities give rise to the four functions listed above, all of which are continuous.

A882. The matrix A has $q^{m(n-m)}$ right inverses over F_q. To see this, we note that, since $AB=I_m$ for some $n\times m$ matrix, B, A must have full rank m. Hence the nullspace of A has dimension $n-m$ and consists of q^{n-m} vectors. Therefore, the right inverses of A are precisely the $(q^{n-m})^m$ matrices obtained from B by adding to each of the m columns of B any one of the q^{n-m} vectors in the nullspace of A.

Editor's Notes (continued from page 224)

Reader Paul Boisvert also commented on *The truel*, by Kilgour and Brams:

> It seems impossible to believe, but the sad truth is that the otherwise interesting article... is marred by a crippling flaw. [The authors] make reference to two Q. Tarantino films involving truels, but neglect to discuss the original, perfect, and still inimitable truel scene in filmic history: the climax of *The Good, the Bad, and the Ugly*. To compare Tarantino's glib, derivative efforts to Sergio Leone's ultimate confrontation among Eastwood, Van Cleef, and Wallach (forming, as they did, a human equilateral triangle inside the circular center of the barren graveyard...) is blasphemy. [Leone's truel] perfectly illustrates the way in which real life always escapes mathematical modeling. The one thing neither [the authors] nor Eli Wallach took into account was that one player might cheat by surreptitiously removing someone else's (Eli's) bullets. As Eli survived—proving that the best strategy may be to have no ammunition whatsoever—The Ugly added a new fillip to the theory, one that I hope the authors will consider in future articles.

PROBLEMS

GEORGE T. GILBERT, *Editor*
Texas Christian University

ZE-LI DOU, KEN RICHARDSON, and SUSAN G. STAPLES, *Assistant Editors*
Texas Christian University

Proposals

To be considered for publication, solutions should be received by November 1, 1998.

1549. *Proposed by K. R. S. Sastry, Bangalore, India.*

Given a positive integer k, prove that for all sufficiently large x, there exist at least k primitive Pythagorean triangles whose sides all have lengths in the interval $[x, 2x]$.

1550. *Proposed by Mihály Bencze, Braşov, Romania.*

Let z_i, $1 \le i \le n$, be complex and let $s_i = z_1 + z_2 + \cdots + z_i$, $1 \le i \le n$. Prove that

$$\sum_{1 \le i \le j \le n} |s_j - z_i| \le \sum_{k=1}^{n} \left((n+1-k)|z_k| + (k-2)|s_k| \right).$$

1551. *Proposed by Howard Morris, Germantown, Tennessee.*

For which values of a is

$$\lim_{n \to \infty} n^2 \ln \frac{\sqrt{2\pi}\,(n+a)^{n+1/2}\, e^{-n-a}}{n!}$$

finite?

1552. *Proposed by Wu Wei Chao, Guang Zhou Normal College, Guang Zhou City, Guang Dong Province, China.*

Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ that satisfy

$$f(x + yf(x)) = f(x) + xf(y)$$

for all x and y.

We invite readers to submit problems believed to be new and appealing to students and teachers of advanced undergraduate mathematics. Proposals must, in general, be accompanied by solutions and by any bibliographical information that will assist the editors and referees. A problem submitted as a Quickie should have an unexpected, succinct solution.

Solutions should be written in a style appropriate for this MAGAZINE. *Each solution should begin on a separate sheet containing the solver's name and full address.*

Solutions and new proposals should be mailed to George T. Gilbert, Problems Editor, Department of Mathematics, Box 298900, Texas Christian University, Fort Worth, TX 76129, or mailed electronically (ideally as a LATEX file) to g.gilbert@tcu.edu. *Readers who use e-mail should also provide an e-mail address.*

1553. *Proposed by Paul Zorn, St. Olaf College, Northfield, Minnesota.*

What complex numbers are the root of some polynomial with positive coefficients?

Quickies

Answers to the Quickies are on page 231

Q880. *Proposed by Ira Rosenholtz, Eastern Illinois University, Charleston, Illinois.*

Show that

$$\sum_{k=0}^{n}(-1)^{n}\binom{n}{k}(n-k)^{n}=n!.$$

Q881. *Proposed by Sergei Ovchinnikov, San Francisco State University, San Francisco, California.*

Describe all continuous functions $f(x, y)$ of two real variables such that $f(x, y) = x$ or $f(x, y) = y$ for all $(x, y) \in \mathbb{R}^2$.

Q882. *Proposed by William P. Wardlow, U.S. Naval Academy, Annapolis, Maryland.*

Let F_q denote a field with q elements. Suppose that A is an $m \times n$ matrix over F_q that has a right inverse over F_q. How many right inverses does A have over F_q?

Solutions

Three Intersecting Cevians **June 1997**

1524. *Proposed by Ted Zerger, Kansas Wesleyan University, Salina, Kansas.*

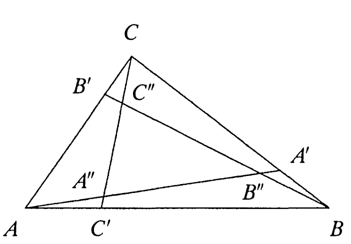
Given ΔABC, let A', B', C' be the points on the sides BC, CA, AB, respectively, such that

$$\frac{BA'}{BC}=\frac{CB'}{AC}=\frac{AC'}{AB}=t, \qquad 0<t<\frac{1}{2}.$$

Let A'', B'', C'' be the points of intersection of AA' and CC', BB' and AA', CC' and BB', respectively. Prove that the ratios

$$AA'' : A''B'' : B''A' = BB'' : B''C'' : C''B' = CC'' : C''A'' : A''C' = t : 1-2t : t^2.$$

(A typographical error in the second display of the original statement has been corrected.)



I. *Solution by Robert L. Young, Osterville, Massachusetts.*

Let 0 be any point in the plane of ΔABC and let X denote the vector from 0 to X. The given imply $\mathbf{A}' = (1-t)\mathbf{B} + t\mathbf{C}, \mathbf{B}' = (1-t)\mathbf{C} + t\mathbf{A}, \mathbf{C}' = (1-t)\mathbf{A} + t\mathbf{B}$, and the existence of scalars λ and μ such that

$$\mathbf{A}'' = (1-\lambda)\mathbf{A} + \lambda\mathbf{A}' = (1-\mu)\mathbf{C} + \mu\mathbf{C}'. \tag{1}$$

Substituting for $\mathbf{A}'$ and $\mathbf{C}'$, we get

$$(1-\lambda)\mathbf{A} + \lambda(1-t)\mathbf{B} + \lambda t\mathbf{C} = \mu(1-t)\mathbf{A} + \mu t\mathbf{B} + (1-\mu)\mathbf{C},$$

or

$$(1-\lambda-\mu+\mu t)(\mathbf{A}-\mathbf{C}) + (\lambda - \lambda t - \mu t)(\mathbf{B}-\mathbf{C}) = \mathbf{0},$$

which implies $\lambda = t/\delta$ and $\mu = (1-t)/\delta$, where $\delta = t^2 - t + 1$. Thus, equation (1) implies $AA'' = \lambda AA' = tAA'/\delta$ and $CA'' = \mu CA' = (1-t)CC'/\delta$. The symmetry of the problem implies $AB'' = (1-t)AA'/\delta$, hence $A''B'' = AB'' - AA'' = (1-2t)AA'/\delta$ and $B''A' = AA' - AB'' = t^2AA'/\delta$. Therefore $AA'' : A''B'' : B''A' = t : 1-2t : t^2$. Similarly, $BB'' : B''C'' : C''B' = CC'' : C''A'' : A''C' = t : 1-2t : t^2$.

II. *Solution by Neela Lakshmanan, University of Scranton, Scranton, Pennsylvania.*

Applying Menelaus' theorem to $\Delta AA'B$ and the transversal CC', we have

$$\frac{AA''}{A''A'} \cdot \frac{A'C}{CB} \cdot \frac{BC'}{C'A} = \frac{AA''}{A''A'} \cdot (1-t) \cdot \frac{1-t}{t} = 1.$$

Thus, $AA''/A''A' = t/(1-t)^2$, and hence $AA''/AA' = t/(1-t+t^2)$. Likewise, by applying Menelaus' theorem to $\Delta AA'C$ and the transversal BB', we obtain $AB''/B''A' = (1-t)/t^2$, hence $AB''/AA' = (1-t)/(1-t+t^2)$. It follows that $A''B''/AA' = (1-2t)/(1-t+t^2)$ and $B''A'/AA' = t^2/(1-t+t^2)$. Therefore,

$$AA'' : A''B'' : B''A' = t : 1-2t : t^2.$$

By symmetrical considerations, we get

$$BB'' : B''C'' : C''B' = CC'' : C''A'' : A''C' = t : 1-2t : t^2.$$

Also solved by J. C. Binz (Switzerland), Mansur Boase (student, England), Sabin Cautis (Canada), Con Amore Problem Group (Denmark), Miguel Amengual Covas (Spain), Daniele Donini (Italy), David Doster, Robert L. Doucette, Ragnar Dybvik (Norway), Milton P. Eisner, Hans Kappus (Switzerland), Atar Sen Mittal, Michael Nathanson, William A. Newcomb, José H. Nieto (Venezuela), Stephen Noltie, P. E. Nüesch (Switzerland), Gao Peng (graduate student), Ron Schryer (professor emeritus), Michael Vowe (Switzerland), David Zhu, and the proposer.

Fixed Ponts of a Bijection of the Symmetric Group **June 1997**

1525. *Proposed by Emeric Deutsch, Polytechnic University, Brooklyn, New York.*

Define a mapping f: $S_n \to S_n$ as follows. Given a permutation π of $\{1, 2, \ldots, n\}$, express it in cycle form, including any fixed elements, such that the smallest entry of each cycle appears last, and the last entries among cycles appear in increasing order. The permutation $f(\pi)$ is then defined by removing all inner parentheses and interpreting the result as the one-line representation of $f(\pi)$. In other words, the ith entry of the line is $f(\pi)(i)$. (For example, expressed in this cycle form, $f((4,6,1)(2)(5\,3)) = (4,2,6,3,1)(5)$.) Characterize those π fixed by f, and determine their cardinality.

Solution by José H. Nieto, Maracaibo, Venezuela.

The permutations fixed by f are those whose cycles are formed by consecutive integers, and their number is 2^{n-1}.

If $\pi \in S_n$ is such that $f(\pi) = \pi$, express it in cycle form as described in the statement of the problem. Its first cycle must be of the form $(a_1, \dots, a_k)$ with $a_k = 1$. Since we have $\pi(a_i) = a_{i+1} = f(\pi)(i+1) = \pi(i+1)$ for $1 \le i \le k-1$, it follows that $a_i = i+1$. Thus the first cycle is $(2, 3, \dots, k, 1)$ (or simply (1) if $k = 1$). If $k < n$, the same reasoning shows that the second cycle is of the form $(k+2, k+3, \dots, l, k+1)$, and that all cycles are formed by consecutive integers in general. Conversely, if $\pi \in S_n$ has this property, a direct verification shows that $f(\pi) = \pi$.

All of these permutations may be generated as follows: write a left parenthesis followed by the numbers from 1 to n, separated by blank spaces, and close the sequence with a right parentheses. Choose any subset of the set of $n-1$ spaces between consecutive integers, and write ")(" in each selected space. Thus we may obtain all the fixed elements of f, expressed as a product of cycles. Its number is 2^{n-1} since this is the number of subsets of a set with $n-1$ elements.

Also solved by Vic Abad, J. C. Binz (Switzerland), David Callan, Con Amore Problem Group (Denmark), Robert L. Doucette, Jerry G. Ianni, Ioana Mihaila, Jean-Claude Ndogmo (South Africa), Allan Pedersen (Denmark), Gao Peng (graduate student), Western Maryland College Problems Group, and the proposer.

Bounded Solutions to a Linear Congruence **June 1997**

1526. *Proposed by Wu Wei Chao, He Nan Normal University, Xin Xiang City, He Nan Province, China.*

Let p be an odd prime number, and let a and b be positive integers with $1 < a < p$. Find the number of ordered pairs (x, y) of positive integers such that p divides $x + ay$ and $x + y < bp$.

Solution by Vic Abad, University of Houston, Houston, Texas.

The number of ordered pairs is $(pb^2 - 3b + 2)/2$.

First, fix x. Because the congruence $x + ay \equiv 0 \pmod{p}$ defines a unique residue class of $y \pmod{p}$, the number of values of y for which p divides $x + ay$ and $0 \le y < bp$ is b. Thus, the number of ordered pairs (x, y) with $0 < x < bp$ and $0 < y < bp$ for which p divides $x + ay$ is $(bp-1)b - (b-1)$. Partition these ordered pairs into three sets according to $x + y < bp$, $x + y > bp$, and $x + y = bp$. The set of (x, y) for which $x + y < bp$ is in one-to-one correspondence with the set of (x', y') for which $x' + y' > bp$ via the map $x' = bp - x$, $y' = bp - y$. For $x + y = bp$, the condition that p divides $x + ay$ is equivalent to p dividing $(a-1)y$, or simply p dividing y. The number of such solutions is $b - 1$. Combining these facts, the number of ordered pairs with $x + y < bp$ is

$$\frac{[(bp-1)b - (b-1)] - (b-1)}{2} = \frac{pb^2 - 3b + 2}{2}.$$

Also solved by J. C. Binz (Switzerland), Mansur Boase (student, England), John Christopher, Con Amore Problem Group (Denmark), Robert L. Doucette, Thomas R. Hagedorn, José H. Nieto (Venezuela), Allan Pedersen (Denmark), and the proposer. There were four incomplete solutions and one incorrect solution.

Computing Terms in a Symmetric Matrix **June 1997**

1527. *Proposed by J. C. Binz, University of Bern, Bern, Switzerland.*

For n a nonnegative integer, let $A_n = (a_{i,k})_{0 \le i,k \le n}$ be the $(n+1) \times (n+1)$ matrix defined by $a_{0,k} = a_{i,0} = 1$ and

$$a_{i,k} = a_{i,k-1} + ima_{i-1,k-1} \qquad (i,k \ge 1).$$

Show that A_n is symmetric, and evaluate $a_{i,k}$.

I. *Solution by Nicholas C. Singer, Annandale, Virginia.*

We show that

$$a_{i,k} = \sum_{j \ge 0} \binom{i}{j}\binom{k}{j} j! m^j$$

by showing that it satisfies the conditions of the problem. This is symmetric in i and k, and equals 1 if i or k is 0. Next,

$$ima_{i-1,k-1} = im \sum_{j \ge 0} \binom{i-1}{j}\binom{k-1}{j} j! m^j = \sum_{j \ge 0} \binom{i-1}{j}\binom{k-1}{j} j! i m^{j+1}$$
$$= \sum_{j \ge 1} \binom{i-1}{j-1}\binom{k-1}{j-1}(j-1)! i m^j = \sum_{j \ge 1} \binom{i}{j}\binom{k-1}{j-1} j! m^j.$$

Thus

$$a_{i,k-1} + ima_{i-1,k-1} = \sum_{j \ge 0} \binom{i}{j}\binom{k-1}{j} j! m^j + \sum_{j \ge 1} \binom{i}{j}\binom{k-1}{j-1} j! m^j$$
$$= 1 + \sum_{j \ge 1} \binom{i}{j} j! m^j \left[\binom{k-1}{j} + \binom{k-1}{j-1}\right]$$
$$= 1 + \sum_{j \ge 1} \binom{i}{j} j! m^j \binom{k}{j} = a_{i,k}.$$

II. *Solution by Western Maryland College Problems Group, Westminster, Maryland.*

We show that

$$a_{ik} = \sum_{j=0}^{\min\{i,k\}} (i)_j (k)_j \frac{m^j}{j!},$$

where $(i)_j := i(i-1) \cdots (i-j+1)$. To arrive at this expression for a_{ik}, we define the generating functions

$$A_k(t) := \sum_{i=0}^{\infty} a_{ik} \frac{t^i}{i!}.$$

The recursion for a_{ik} leads to

$$A_k(t) = (1+mt) A_{k-1}(t) = \cdots = (1+mt)^k A_0(t)$$
$$= \sum_{j=0}^{k} \binom{k}{j} m^j t^j \cdot \sum_{n=0}^{\infty} \frac{t^n}{n!}.$$

The coefficient of t^i can now be read off and after multiplying by $i!$ we recover the stated expression for a_{ik}. The symmetry follows for free.

Also solved by Vic Abad, Robert A. Agnew, Dale R. Buske, David Callan, Con Amore Problem Group (Denmark), Robert L. Doucette, Bassem B. Ghalayini and Ajaj A. Tarabay (Lebanon), José H. Nieto (Venezuela), Allan Pedersen (Denmark), Gao Peng (graduate student), Heinz-Jürgen Seiffert (Germany), Michael Vowe (Switzerland), and the proposer.

Inequalities in a Convex n-gon **June 1997**

1528. *Proposed by Florin S. Pirvănescu, Slatina, Romania.*

Let M be a point in the interior of convex polygon $A_1A_2\ldots A_n$. If d_k is the distance from M to A_kA_{k+1} $(A_{n+1}=A_1)$, show that

$$(d_1+d_2)(d_2+d_3)\cdots(d_n+d_1)\le 2^n\cos^n\frac{\pi}{n}\cdot MA_1\cdot MA_2\cdot\cdots\cdot MA_n,$$

and determine when equality holds.

Solution by Heinz-Jürgen Seiffert, Berlin, Germany.

Set $\alpha_k=\angle MA_kA_{k-1}$ and $\beta_k=\angle MA_kA_{k+1}$, $k=1,\ldots,n$ $(A_0=A_n)$. We have $d_k=MA_{k+1}\sin\alpha_{k+1}=MA_k\sin\beta_k$, which implies

$$d_{k-1}+d_k=MA_k(\sin\alpha_k+\sin\beta_k)=2MA_k\sin\frac{\alpha_k+\beta_k}{2}\cos\frac{\alpha_k-\beta_k}{2},\ k=1,\ldots,n.$$

It follows that

$$\prod_{k=1}^{n}(d_{k-1}+d_k)=2^n\prod_{k=1}^{n}\left(MA_k\cdot\sin\frac{\alpha_k+\beta_k}{2}\cos\frac{\alpha_k-\beta_k}{2}\right).$$

Since the sine is strictly increasing and concave on $(0,\pi/2)$, from the arithmetic mean-geometric mean inequality and Jensen's Inequality we get

$$\prod_{k=1}^{n}\sin\frac{\alpha_k+\beta_k}{2}\le\sin^n\left(\sum_{k=1}^{n}\frac{\alpha_k+\beta_k}{2n}\right)=\cos^n(\pi/n),$$

with equality if and only if $\alpha_1+\beta_2=\alpha_2+\beta_2=\cdots=\alpha_n+\beta_n$, where we have used $\sum_{k=1}^{n}(\alpha_k+\beta_k)=(n-2)\pi$. Moreover, we have

$$\prod_{k=1}^{n}\cos\frac{\alpha_k-\beta_k}{2}\le 1,$$

with equality if and only if $\alpha_k=\beta_k$ for $k=1,\ldots,n$. The desired inequality follows.

Clearly, there is equality if $A_1A_2\ldots A_n$ is regular and M is its center. Conversely, if equality holds, then from above we have $\alpha_k=\beta_k=(1/2-1/n)\pi$ for $k=1,\ldots,n$. It then easily follows that $MA_1=MA_2=\cdots=MA_n$, and further that $A_1A_2\ldots A_n$ is equilateral. Therefore, the convex polygon $A_1A_2\ldots A_n$ must be regular. Now, it is easily seen that M must be the center.

Comment. Murray Klamkin observed that the result follows from the stronger inequality with d_k redefined to be the length of the angle bisector of $\angle A_kMA_{k+1}$, referring us to D. S. Mitrinovic, J. E. Pecaric, and V. Volenic, *Recent Advances in Geometric Inequalities*, p. 423.

Also solved by Mansur Boase (student, England), Con Amore Problem Group (Denmark), Robert L. Doucette, Lorraine L. Foster and Tung-Po Lin, Murray S. Klamkin, Can A. Minh (graduate student), José H. Nieto (Venezuela), Stephen Noltie, Allan Pedersen (Denmark), Gao Peng (graduate student), Achilleas Sinefakopoulos (student, Greece), Michael Vowe (Switzerland), Robert L. Young, and the proposer.

Answers

Solutions to the Quickies on page 226.

A880. The right-hand side is clearly the number of one-to-one functions from an n-element set to itself.

Since there are $(n-k)^n$ functions from an n-element set to an $(n-k)$-element set, and an n-element set has $\binom{n}{k}$ subsets with $(n-k)$ elements, the left-hand side is the number of onto functions from an n-element set to itself (by the inclusion-exclusion principle).

Finally, since the one-to-one functions and the onto functions from an n-element set to itself are identical, equality follows.

A881. The only four such functions are

$$f(x,y)=x, f(x,y)=y, f(x,y)=\min\{x,y\}, f(x,y)=\max\{x,y\}.$$

Clearly $f(x,y)=x=y$ on the line $x=y$. Let $A=\{(x,y): f(x,y)=x\}$. Then A is a closed set and its complement A^c is the open set $A^c=\{(x,y): f(x,y)=y, x\neq y\}$. Similarly, $B=\{(x,y): f(x,y)=y\}$ is closed and $B^c=\{(x,y): f(x,y)=x, x\neq y\}$ is open. The open half-plane $U=\{(x,y): x>y\}$ is a connected set. Since U is the disjoint union of the open sets $U\cap A^c$ and $U\cap B^c$, either $U\cap A^c=\varnothing$ or $U\cap B^c=\varnothing$. Thus, either $f(x,y)=x$ or $f(x,y)=y$ on U. Similarly, either $f(x,y)=x$ or $f(x,y)=y$ on $\{(x,y): x<y\}$. The four combinations of possibilities give rise to the four functions listed above, all of which are continuous.

A882. The matrix A has $q^{m(n-m)}$ right inverses over F_q. To see this, we note that, since $AB=I_m$ for some $n\times m$ matrix, B, A must have full rank m. Hence the nullspace of A has dimension $n-m$ and consists of q^{n-m} vectors. Therefore, the right inverses of A are precisely the $(q^{n-m})^m$ matrices obtained from B by adding to each of the m columns of B any one of the q^{n-m} vectors in the nullspace of A.

Editor's Notes (*continued from page 224*)

Reader Paul Boisvert also commented on *The truel*, by Kilgour and Brams:

> It seems impossible to believe, but the sad truth is that the otherwise interesting article... is marred by a crippling flaw. [The authors] make reference to two Q. Tarantino films involving truels, but neglect to discuss the original, perfect, and still inimitable truel scene in filmic history: the climax of *The Good, the Bad, and the Ugly*. To compare Tarantino's glib, derivative efforts to Sergio Leone's ultimate confrontation among Eastwood, Van Cleef, and Wallach (forming, as they did, a human equilateral triangle inside the circular center of the barren graveyard...) is blasphemy. [Leone's truel] perfectly illustrates the way in which real life always escapes mathematical modeling. The one thing neither [the authors] nor Eli Wallach took into account was that one player might cheat by surreptitiously removing someone else's (Eli's) bullets. As Eli survived—proving that the best strategy may be to have no ammunition whatsoever—The Ugly added a new fillip to the theory, one that I hope the authors will consider in future articles.

REVIEWS

PAUL J. CAMPBELL, *editor*
Beloit College
1997–98: Universität Augsburg,
Germany

Assistant Editor: Eric S. Rosenthal, West Orange, NJ. Articles and books are selected for this section to call attention to interesting mathematical exposition that occurs outside the mainstream of mathematics literature. Readers are invited to suggest items for review to the editors.

Wells, Charles, *The Handbook of Mathematical Discourse*, Version 0.71, March 1998, `http://www-math.cwru.edu/~cfw2/abouthbk.htm` , or email the author (`cfw2@po.cwru.edu`) for a printed copy; x + 135 pp. Communicating mathematics: Useful ideas from computer science, *American Mathematical Monthly* 102 (1995) 397–408. Bagchi, Atish, and Charles Wells, On the communication of mathematical reasoning, *Primus* 8 (1) (March 1998) 15–27. Bagchi, Atish, and Charles Wells, Varieties of mathematical prose. The papers are available at `www.cwru.edu/artsci/math/wells/pub/papers.html` .

Is mathematics a foreign—or even alien—language for students? Many think so, and author Wells concurs. He and colleague Bagchi have begun a long-overdue investigation of the *mathematical register*, the choice of English and symbolism used in communication of mathematics. Wells's book-in-the-making is a dictionary of rhetorical terms and compilation of their usage in mathematical exposition—an attempt to make mathematicians aware of how they talk and write, particularly the ways that vary from usage by others. Wells tries to put a name to each kind of usage, employing terms from standard rhetoric (e.g., enthymeme), from mathematical education (malrule), and from his own colorful coinage (existential bigamy, jump the fence). Students too should find the book useful. Wells solicits contributions of citations and suggestions from readers. While the book is descriptive, the papers by Wells and with colleague Bagchi make specific normative recommendations about oral and written mathematical exposition. (Fortunately, the book and papers are available electronically in dvi, PS, and pdf formats. Too many other documents on the Web are available only by inconvenient multiple downloads in HTML, a few pages at a time, with equations as individual figures that print ugly.)

Maligranda, Lech, Why Hölder's inequality should be called Rogers' inequality, *Mathematical Inequalities and Applications* 1 (1) (January 1998) 69–83.

The inequality

$$\sum_{k=1}^{n} a_k b_k \leq \left(\sum_{k=1}^{n} a_k^p\right)^{1/p} \left(\sum_{k=1}^{n} b_k^q\right)^{1/q},$$

true for $a_k, b_k > 0$, $p > 1$, and $1/p + 1/q = 1$, was proved in slightly different form by Leonard J. Rogers (of Rogers–Ramanujan identities fame) in 1888, a year before Otto Hölder proved a more general result. Although Hölder cited Rogers, subsequent authors named the inequality after Hölder, who had published in a more accessible journal. Author Maligranda documents the history and mathematics involved and urges that the result be renamed the *Rogers–Hölder inequality*. Perhaps some enterprising soul will start a Web site to collect together all such suggested revisions of history of mathematics, despite Fejér's caution that "the history of mathematics serves to prove that nobody has discovered anything: there was always somebody who knew it before." (Thanks to author Maligranda at `lech@sm.luth.se` for volunteering a reprint from the first issue of this new journal.)

Forbes, Tony, Ten Primes: A search for ten consecutive primes in arithmetic progression: Success!!!, `http://www.ltkz.demon.co.uk/ar2/10primes.htm` .

In the April 1998 issue of THIS MAGAZINE, I reported on the discovery of nine consecutive primes in arithmetic progression, suggesting that you could be confident that the record would stand for some time. Wrong! However, the team believes that finding a sequence of eleven is "far too difficult," in part because the minimum progression gap becomes 2,310 instead of 210. "We need a new idea, or a trillion-fold increase in computer speeds. So we expect the Ten Primes record to stand for a long time to come." This time, it's their prediction, not mine.

Avnir, David, Ofer Biham, Daniel Lidar, and Ofer Malcai, Is the geometry of nature fractal?, *Science* 279 (2 January 1998) 39–40.

Answer: Maybe no, but we will keep saying that it is. "Do power laws that are limited in range represent fractals? Is it justified to term them as such?" The authors refer to calculating a fractal dimension D from a relation of the form $P = kr^{f(D)}$, where $f(D)$ is a simple function of D. A fit of data to such a law does not imply fractality over many orders of magnitude; in few reports in the physics literature does a fit span more than two orders of magnitude and in no case more than three. Also, few such reports have any theoretical backing. However, such laws can be useful on their own, without the trendy "fractal" label. Moreover, "[s]everal key processes involving equilibrium-critical phenomena (in magnets, liquids, percolations, and phase transitions, for example) and some nonequilibrium growth models (such as aggregation) are backed by intrinsically scale-free theories and lead therefore to power-law scaling behavior on all scales."

Cipra, Barry, Proving a link between logic and origami, *Science* 279 (6 February 1998) 804–805.

Origami is the art of folding shapes (e.g., peace cranes) from a square of paper. It involves several skills: devising a pattern and sequence of creases to create a shape, discerning from the crease lines the order of folding (I find this nontrivial even for roadmaps), and predicting properties of the folded object from the crease pattern (e.g., a roadmap is supposed to fold flat). The question of whether a pattern can be folded flat turns out to be NP-complete. Barry Hayes (Placeware Inc., Mountain View, Calif.) and Marshall Bern (Xerox Palo Alto Research Center) translated logical expressions into crease patterns and showed that the flat-folding problem is equivalent to the NP-hard problem called *not-all-true 3-SAT*. This problem is, given a sentence in propositional logic consisting of three-variable clauses, in each of which not all three variables are assigned True, determine if there is a truth-assignment that makes the sentence true.

Cipra, Barry, Sieving prime numbers from thin ore, *Science* 279 (2 January 1998) 31.

A mathematical sieve is an algorithm to eliminate non-primes from a sequence. For example, the Euclidean sieve, applied to 2 through 100, first eliminates all multiples of 2, then all remaining multiples of 3, and so on. Stopping after eliminating multiples of 5 gives an estimate of 28 for the number of primes; the sieve fails to eliminate only 49, 77, and 91. John Friedlander (University of Toronto) and Henryk Iwaniec (Rutgers University) have refined the *asymptotic sieve*, developed by E. Bombieri in the 1970s, to show that numbers of the form $a^2 + b^4$—a sequence with asymptotic density zero—include infinitely many primes. The peculiar form of $a^2 + b^4$ facilitates use of Gaussian integers (of the form $a + b\sqrt{-1}$) and theory from algebraic number theory. Tantalizing in their simplicity are such open problems as whether numbers of the form $n^2 + 1$ include an infinite number of primes, or if each interval between consecutive squares must contain a prime.

NEWS AND LETTERS

Twenty-Sixth Annual USA Mathematical Olympiad – Problems and Solutions

1. Let $p_1, p_2, p_3, \ldots$ be the prime numbers listed in increasing order, and let x_0 be a real number between 0 and 1. For positive integers k, define

$$x_k = \begin{cases} 0 & \text{if } x_{k-1} = 0, \\ \left\{ \dfrac{p_k}{x_{k-1}} \right\} & \text{if } x_{k-1} \neq 0, \end{cases}$$

where $\{x\}$ denotes the fractional part of x. (The fractional part of x is given by $x - \lfloor x \rfloor$ where $\lfloor x \rfloor$ is the greatest integer less than or equal to x.) Find, with proof, all x_0 satisfying $0 < x_0 < 1$ for which the sequence $x_0, x_1, x_2, \ldots$ eventually becomes 0.

Solution. The sequence eventually becomes 0 if and only if x_0 is a rational number.

First we prove that, for $k \geq 1$, every rational term x_k has a rational predecessor x_{k-1}. Suppose x_k is rational. If $x_k = 0$ then either $x_{k-1} = 0$ or p_k/x_{k-1} is a positive integer; either way, x_{k-1} is rational. If x_k is rational and nonzero, then the relation

$$x_k = \left\{ \frac{p_k}{x_{k-1}} \right\} = \frac{p_k}{x_{k-1}} - \left\lfloor \frac{p_k}{x_{k-1}} \right\rfloor \quad \text{yields} \quad x_{k-1} = \frac{p_k}{x_k + \left\lfloor \dfrac{p_k}{x_{k-1}} \right\rfloor},$$

which shows that x_{k-1} is rational. Since every rational term x_k with $k \geq 1$ has a rational predecessor, it follows by induction that, if x_k is rational for some k, then x_0 is rational. In particular, if the sequence eventually becomes 0, then x_0 is rational.

To prove the converse, observe that if $x_{k-1} = m/n$ with $0 < m < n$, then $x_k = r/m$, where r is the remainder that results from dividing np_k by m. Hence the denominator of each nonzero term is strictly less than the denominator of the term before. In particular, the number of nonzero terms in the sequence cannot exceed the denominator of x_0.

Note that the above argument applies to any sequence $\{p_k\}$ of positive integers, not just the sequence of primes.

2. Let ABC be a triangle, and draw isosceles triangles BCD, CAE, ABF externally to ABC, with BC, CA, AB as their respective bases. Prove that the lines through A, B, C perpendicular to the lines $\overleftrightarrow{EF}, \overleftrightarrow{FD}, \overleftrightarrow{DE}$, respectively, are concurrent.

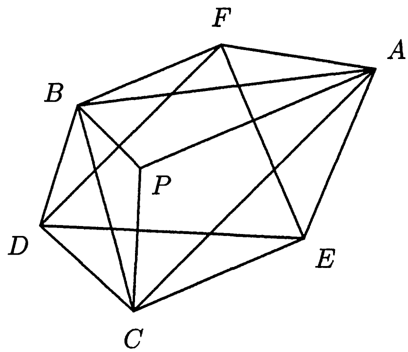
Solution. We first show that for any four points W, X, Y, Z in the plane, the lines WX and YZ are perpendicular if and only if

$$WY^2 - WZ^2 = XY^2 - XZ^2. \qquad (*)$$

To prove this, introduce Cartesian coordinates such that $W = (0,0)$, $X = (1,0)$, $Y = (x_1, y_1)$, and $Z = (x_2, y_2)$. Then (*) becomes

$$x_1^2 + y_1^2 - x_2^2 - y_2^2 = (x_1 - 1)^2 + y_1^2 - (x_2 - 1)^2 - y_2^2,$$

which upon cancellation yields $x_1 = x_2$. This is true if and only if line YZ is perpendicular to the x-axis WX.



If P is the intersection of the perpendiculars from B and C to lines FD and DE, respectively, then the fact noted above yields

$$PF^2 - PD^2 = BF^2 - BD^2, \quad \text{and} \quad PD^2 - PE^2 = CD^2 - CE^2.$$

From the given isosceles triangles, we have $BF = AF$, $BD = CD$, and $CE = AE$. Therefore

$$PF^2 - PE^2 = AF^2 - AE^2.$$

Hence line PA is also perpendicular to line EF, which completes the proof.

Second Solution: Let $\mathcal{C}_1$ be the circle with center D and radius BD, $\mathcal{C}_2$ the circle with center E and radius CE, and $\mathcal{C}_3$ the circle of center F and radius AF. The line through A and perpendicular to EF is the radical axis of circles $\mathcal{C}_2$ and $\mathcal{C}_3$, the line through B and perpendicular to DF is the radical axis of circles $\mathcal{C}_1$ and $\mathcal{C}_3$, and the line through C and perpendicular to DE is the radical axis of circles $\mathcal{C}_1$ and $\mathcal{C}_2$. The result follows because these three radical axes meet at the radical center of the three circles.

3. Prove that for any integer n, there exists a unique polynomial Q with coefficients in $\{0, 1, \ldots, 9\}$ such that $Q(-2) = Q(-5) = n$.

Solution. First suppose there exists a polynomial Q with coefficients in $\{0, 1, \ldots, 9\}$ such that $Q(-2) = Q(-5) = n$. We shall prove that this polynomial is unique. By the Factor Theorem, we can write $Q(x) = P(x)R(x) + n$ where $P(x) = (x + 2)(x + 5) = x^2 + 7x + 10$ and $R(x) = r_0 + r_1 x + r_2 x^2 + \cdots$ is a polynomial. Then $r_0, r_1, r_2, \ldots$ are integers such that

$$10r_0 + n \in \{0, 1, \ldots, 9\}, \qquad 10r_k + 7r_{k-1} + r_{k-2} \in \{0, 1, \ldots, 9\}, \quad k \geq 1 \qquad (*)$$

(with the understanding that $r_{-1} = 0$). For each k, (*) uniquely determines r_k once r_j is known for all $j < k$. Uniqueness of R, and therefore of Q, follows.

Existence will follow from the fact that for the unique sequence $\{r_k\}$ satisfying (*), there exists some N such that $r_k = 0$ for all $k \geq N$. First note that $\{r_k\}$ is bounded, since $|r_0|, |r_1| \leq B$ and $B \geq 9$ imply $|r_k| \leq B$ for all k. This follows by induction, using $10|r_k| \leq 7|r_{k-1}| + |r_{k-2}| + 9 \leq 10B$. More specifically, if $r_i \leq M$ for $i = k-1,\ k-2$, then

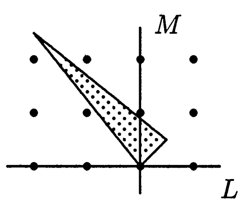
$$r_k \geq -\frac{7r_{k-1}}{10} - \frac{r_{k-2}}{10} \geq -\frac{4M}{5},$$

while if $r_i \geq L$ for $i = k-1,\ k-2$, then

$$r_k \leq -\frac{7r_{k-1}}{10} - \frac{r_{k-2}}{10} + \frac{9}{10} \leq -\frac{4L}{10} + \frac{9}{10}.$$

Since the sequence $\{r_k\}$ is bounded, we can define $L_k = \min\{r_k, r_{k-1}, \dots\}$ and $M_k = \max\{r_k, r_{k+1}, \dots\}$. Clearly $L_k \leq L_{k+1}$ and $M_k \geq M_{k+1}$ for all k.

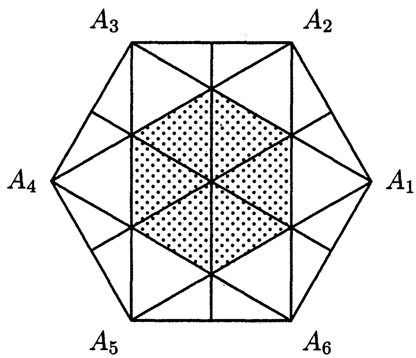
Since $L_k \leq M_k$ for all k, the non-decreasing sequence $\{L_k\}$ must stop increasing eventually, and, similarly, the non-increasing sequence $\{M_k\}$ must stop decreasing. In other words, there exist L, M, N such that $L_k = L$ and $M_k = M$ for all $k \geq N$. Certainly $L \leq M$, and $M \geq 0$, since no three consecutive terms in $\{r_k\}$ can be negative, but the above arguments also imply $L \geq -4M/5$ and $M \leq -4L/5 + 9/10$. A quick sketch (shown below) shows that the set of real pairs (L, M) satisfying these conditions is a closed triangular region containing no lattice points other than $(0, 0)$. It follows that $r_k = 0$ for all $k \geq N$, proving existence.



4. To *clip* a convex n-gon means to choose a pair of consecutive sides AB, BC and to replace them by the three segments AM, MN, and NC, where M is the midpoint of AB and N is the midpoint of BC. In other words, one cuts off the triangle MBN to obtain a convex $(n+1)$-gon. A regular hexagon $\mathcal{P}_6$ of area 1 is clipped to obtain a heptagon $\mathcal{P}_7$. Then $\mathcal{P}_7$ is clipped (in one of the seven possible ways) to obtain an octagon $\mathcal{P}_8$, and so on. Prove that no matter how the clippings are done, the area of $\mathcal{P}_n$ is greater than 1/3, for all $n \geq 6$.

Solution. The key observation is that for any side S of $\mathcal{P}_6$, there is some subsegment of S that is a side of $\mathcal{P}_n$. (This is easily proved by induction on n.) Thus $\mathcal{P}_n$ has a vertex on each side of $\mathcal{P}_6$. Since $\mathcal{P}_n$ is convex, it contains a hexagon $\mathcal{Q}$ with (at least) one vertex on each side of $\mathcal{P}_6$. (The hexagon may be degenerate, as some of its vertices may coincide.)

Let $\mathcal{P}_6 = A_1A_2A_3A_4A_5A_6$, and let $\mathcal{Q} = B_1B_2B_3B_4B_5B_6$, with B_i on A_iA_{i+1} (indices are considered modulo 6). The side B_iB_{i+1} of $\mathcal{Q}$ is entirely contained in triangle $A_iA_{i+1}A_{i+2}$, so $\mathcal{Q}$ encloses the smaller regular hexagon $\mathcal{R}$ (shaded in the diagram below) whose sides are the central thirds of the segments A_iA_{i+2}, $1 \leq i \leq 6$. The area of $\mathcal{R}$ is 1/3, as can be seen from the fact that its side length is $1/\sqrt{3}$ times the side length of $\mathcal{P}_6$, or from a dissection argument (count the small equilateral triangles and halves thereof in the diagram below). Thus Area$(\mathcal{P}_n) \geq$ Area$(\mathcal{Q}) \geq$ Area$(\mathcal{R}) = 1/3$. We obtain strict inequality by observing that $\mathcal{P}_n$ is strictly larger that $\mathcal{Q}$: if $n = 6$, this is obvious; if $n > 6$, then $\mathcal{P}_n$ cannot equal $\mathcal{Q}$ because $\mathcal{P}_n$ has more sides.



Note. With a little more work, one could improve 1/3 to 1/2. The minimal area of a hexagon $\mathcal{Q}$ with one vertex on each side of $\mathcal{P}_6$ is in fact 1/2, attained when the vertices of $\mathcal{Q}$ coincide in pairs at every other vertex of $\mathcal{P}_6$, so the hexagon $\mathcal{Q}$ degenerates into an equilateral triangle. If the conditions of the problem were changed so that the "cut-points" could be anywhere within adjacent segments instead of just at the midpoints, then the best possible bound would be 1/2.

5. Prove that, for all positive real numbers a, b, c,

$$(a^3 + b^3 + abc)^{-1} + (b^3 + c^3 + abc)^{-1} + (c^3 + a^3 + abc)^{-1} \leq (abc)^{-1}.$$

Solution. The inequality $(a - b)(a^2 - b^2) \geq 0$ implies $a^3 + b^3 \geq ab(a + b)$, so

$$\frac{1}{a^3 + b^3 + abc} \leq \frac{1}{ab(a + b) + abc} = \frac{c}{abc(a + b + c)}.$$

Similarly

$$\frac{1}{b^3 + c^3 + abc} \leq \frac{1}{bc(b + c) + abc} = \frac{a}{abc(a + b + c)},$$

and

$$\frac{1}{c^3 + a^3 + abc} \leq \frac{1}{ca(c + a) + abc} = \frac{b}{abc(a + b + c)}.$$

Thus

$$\frac{1}{a^3 + b^3 + abc} + \frac{1}{b^3 + c^3 + abc} + \frac{1}{c^3 + a^3 + abc} \leq \frac{a + b + c}{abc(a + b + c)} = \frac{1}{abc}.$$

6. Suppose the sequence of nonnegative integers $a_1, a_2, \ldots, a_{1997}$ satisfies

$$a_i + a_j \leq a_{i+j} \leq a_i + a_j + 1$$

for all $i, j \geq 1$ with $i + j \leq 1997$. Show that there exists a real number x such that $a_n = \lfloor nx \rfloor$ (the greatest integer $\leq nx$) for all $1 \leq n \leq 1997$.

Solution. Any x that lies in all of the half-open intervals

$$I_n = \left[\frac{a_n}{n}, \frac{a_n + 1}{n}\right), \qquad n = 1, 2, \ldots, 1997$$

will have the desired property. Let

$$L = \max_{1 \leq n \leq 1997} \frac{a_n}{n} = \frac{a_p}{p} \quad \text{and} \quad U = \min_{1 \leq n \leq 1997} \frac{a_n + 1}{n} = \frac{a_q + 1}{q}.$$

We shall prove that $\frac{a_n}{n} < \frac{a_m + 1}{m}$, or, equivalently,

$$m a_n < n(a_m + 1) \tag{$*$}$$

for all m, n ranging from 1 to 1997. Then $L < U$, since $L \geq U$ implies that $(*)$ is violated when $n = p$ and $m = q$. Any point x in $[L, U)$ has the desired property.

We prove $(*)$ for all m, n ranging from 1 to 1997 by strong induction. The base case $m = n = 1$ is trivial. The induction step splits into three cases. If $m = n$, then $(*)$ certainly holds. If $m > n$, then the induction hypothesis gives $(m-n)a_n < n(a_{m-n} + 1)$, and adding $n(a_{m-n} + a_n) \leq n a_m$ yields $(*)$. If $m < n$, then the induction hypothesis yields $m a_{n-m} < (n-m)(a_m + 1)$, and adding $m a_n \leq m(a_m + a_{n-m} + 1)$ gives $(*)$.

Thirty-Eighth Annual International Mathematical Olympiad – Problems

1. In the plane the points with integer coordinates are the vertices of unit squares. The squares are colored alternately black and white (as on a chessboard). For any pair of positive integers m and n, consider a right-angled triangle whose vertices have integer coordinates and whose legs, of lengths m and n, lie along edges of the squares. Let S_1 be the total area of the black part of the triangle and S_2 the total area of the white part. Let

$$f(m, n) = |S_1 - S_2|.$$

(a) Calculate $f(m, n)$ for all positive integers m and n which are either both even or both odd.

(b) Prove that $f(m, n) \leq \frac{1}{2}\max\{m, n\}$ for all m and n.

(c) Show that there is no constant C such that $f(m, n) < C$ for all m and n.

2. Angle A is the smallest angle in triangle ABC. Points B and C divide the circumcircle of the triangle into two arcs. Let U be an interior point of the arc between B and C which does not contain A. The perpendicular bisectors of AB and AC meet the line AU at V and W, respectively. Lines BV and CW meet at T. Show that

$$AU = TB + TC.$$

3. Let $x_1, x_2, \ldots, x_n$ be real numbers satisfying the conditions: $|x_1 + x_2 + \cdots + x_n| = 1$ and $|x_i| \leq (n+1)/2$ for $i = 1, 2, \ldots, n$. Show that there exists a permutation $y_1, y_2, \ldots, y_n$ of $x_1, x_2, \ldots, x_n$ such that

$$|y_1 + 2y_2 + \cdots + n y_n| \leq \frac{n+1}{2}.$$

4. An $n \times n$ matrix (square array) whose entries come from the set $S = \{1, 2, \ldots, 2n-1\}$ is called a *silver* matrix if, for each $i = 1, 2, \ldots, n$, the i^{th} row and i^{th} column together contain all elements of S. Show that

(a) there is no silver matrix for $n = 1997$;

(b) silver matrices exist for infinitely many values of n.

5. Find all pairs (a, b) of positive integers that satisfy the equation $a^{b^2} = b^a$.

6. For each positive integer n, let $f(n)$ denote the number of ways of representing n as a sum of powers of 2 with nonnegative integer exponents. Representations which differ only in the ordering of their summands are considered to be the same. For instance, $f(4) = 4$ because the number 4 can be represented in the following four ways: 4; $2+2$; $2+1+1$; $1+1+1+1$. Prove that for every integer $n \geq 3$,

$$2^{n^2/4} < f(2^n) < 2^{n^2/2}.$$

Notes

The top eight students on the 1997 USAMO were (in alphabetical order):

Carl J. Bosley	Topeka, KS
Li-Chung Chen	Cupertino, CA
John J. Clyde	New Plymouth, ID
Nathan G. Curtis	Alexandria, VA
Kevin D. Lacker	Cincinnati, OH
Davesh Maulik	Roslyn Heights, NY
Josh P. Nichols-Barrer	Newton Center, MA
Daniel P. Stronger	New York, NY

Josh Nichols-Barrer was the winner of the Greitzer-Klamkin award, given to the top scorer on the USAMO. Members of the USA team at the 1997 IMO (Mar del Plata, Argentina) were Carl Bosley, Li-Chung Chen, John Clyde, Nathan Curtis, Josh Nichols-Barrer, and Daniel Stronger. Bosley and Curtis both received gold medals, while Chen, Clyde, Nichols-Barrer, and Stronger received silver medals. In terms of total score, the highest ranking of the eighty-two participating teams were as follows:

China	223	Ukraine	195
Hungary	219	Bulgaria	191
Iran	217	Romania	191
United States	204	Australia	187
Russia	204	Vietnam	183

The 1997 USA Mathematical Olympiad was prepared by Titu Andreescu, Elgin Johnston, Jim Propp, Cecil Rousseau (chair), Alexander Soifer, Richard Stong, and Paul Zeitz. The training program to prepare the USA team for the IMO (the Mathematical Olympiad Summer Program) was held at the University of Nebraska, Lincoln, NE. Titu Andreescu (Director), Fan Chung, Zuming Feng, Razvan Gelca, Elgin Johnston, and Kiran Kedlaya served as instructors, assisted by Jeremy Bem and Jonathan Weinstein.

The booklet *Mathematical Olympiads 1997* presents additional solutions to problems on the 26th USAMO and solutions to the 38th IMO. Such a booklet has been published every year since 1976. Copies are $5.00 for each year 1976–1997. They are available from:

> Dr. Walter Mientka, Department of Mathematics, University of Nebraska, Lincoln, NE 68588-0658.

The USA Mathematical Olympiad, participation of the US team in the International Mathematical Olympiad, and the sequence of examinations leading to qualification for these olympiads are under the administration of the M.A.A. Committee on American Mathematical Competitions, and these activities are sponsored by eight organizations of professional mathematicians. For further information about this sequence of examinations, contact the Executive Director of the Committee, Professor Mientka, at the above address.

This report was prepared by Titu Andreescu, Elgin Johnston, and Cecil Rousseau.

Letter to the Editor

Dear Editor:

Mark Krusemeyer's recursively defined bijection between two manifestations of the Catalan numbers (*A parenthetical note (to a paper of Guy)*, this MAGAZINE, October 1996, pp. 257–260) also has a simple explicit description as follows. Given a CG-arrangement (legal arrangement of pairs of empty parentheses), first delete the leftmost left parenthesis, change each right parenthesis to a letter (x, say), and add a letter at the end. Then insert right parentheses anew (in the only way possible) to produce a legal bracketing of the letters. For example, $()()(()) \to x(x((xxx \to x(x((xx)x))$ agrees with $\lceil\rceil\lceil\rceil\lceil\lceil\rceil\rceil \to a\lfloor b\lfloor\lfloor cd\rfloor e\rfloor\rfloor$ in Krusemeyer's table for $n=4$. Expressed this way, the bijection's inverse becomes obvious. (This construction is implicit in an argument on page 54 of *Advanced Combinatorics*, Louis Comtet, D. Reidel, Boston, 1974.)

It is fairly easy to show the construction works: the first step either yields a mere xx, and no parenthesizing is needed, or else there must be at least one occurrence of $(xx$, and a right parenthesis must be inserted immediately after it; then we treat this (xx) as a single x and proceed by induction.

That our mapping (say ϕ) agrees with Krusemeyer's F can also be established by induction. The base cases may be verified directly. For the induction step, it is helpful to let ψ denote the initial mapping (before the right parentheses, or right floor symbols in Krusemeyer's notation, are inserted) and to consider four cases for $\alpha = \lceil\beta\rceil\gamma$, according as the CG-arrangements β and γ are empty or not. For example, if $\beta \neq \emptyset, \gamma \neq \emptyset$, then α looks like $\lceil\underbrace{\lceil * * **\rceil}_{\beta}\rceil\,\underbrace{\lceil **\rceil}_{\gamma}$ where the asterisks denote any number (possibly 0) of ceiling symbols. By definition, $\psi\alpha$ is $\lfloor\underbrace{* * * * x\,x}_{\psi\beta}\lfloor\underbrace{* * x\,x}_{\psi\gamma}$ where the asterisks are now letters or left floors. Also, $\psi\beta$ and $\psi\gamma$ appear in $\psi\alpha$ as indicated. Finally, inserting the right floors (in the only way possible) yields $\phi\alpha = \lfloor\phi\beta\rfloor\lfloor\phi\gamma\rfloor$ and the induction step follows in this case. The other cases are similar.

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